Public Finance
and the Balance Sheet of the Private Sector*

Hans Gersbach† Jean-Charles Rochet‡
Ernst-Ludwig von Thadden§

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Abstract

This paper characterizes optimal fiscal policy in a simple endogenous growth model under endogenously incomplete financial markets and idiosyncratic production risk. Entrepreneurs do not obtain sufficient funding because of informational frictions. In the second-best optimum, they issue continuously traded short-term debt. The government funds public expenditure by unrestricted debt and taxes, maximizing a weighted sum of the welfare of entrepreneurs and households. We show how any constrained Pareto-optimal allocation can be decentralized as a competitive equilibrium by issuing straight public debt combined with heterogeneous linear wealth taxation. The model allows for explicit comparative statics of public and private debt, growth, interest, and the wealth distribution.

Keywords: Incomplete Financial Markets, Outside Finance, Heterogeneous Agents, Debt, Taxes, Interest, Growth, Inequality

JEL Classification: D31, D43, D52, D63, E21, E44, E62.

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†KOF Swiss Economic Institute, ETH Zurich and CEPR Email: hgersbach@ethz.ch.

‡Toulouse School of Economics, Email: jeancharles.rochet@gmail.com.

§Universität Mannheim, CEPR, and ECGI. Email: vthadden@uni-mannheim.de.

1 Introduction

This paper studies fiscal policy in a simple endogenous growth model in the presence of incomplete financial markets, heterogeneous agents with conflicting interests, and non-insurable idiosyncratic production risk. Our approach is different from previous work because we ask why markets are incomplete in the first place and what this implies for firms' balance sheets and investment. In such a framework, we ask how a government should finance public spending and optimally coordinate private investment and growth. More specifically, we ask: how much of public spending should be financed through taxes compared to debt? What are the distributional consequences of issuing public debt, and how can one engineer Pareto improvements? Given the incentive problems of smoothing individual shocks, will optimal public policy increase inequality and possibly even lead to massive disparities? How do these choices depend on the weight of different interests in the economy, and how do they affect equilibrium interest rates, savings, growth, and fiscal sustainability in the long run?

To answer these questions, we develop an analytically tractable dynamic macroeconomic model along classical lines of Merton (1971), Dumas (1989), and more recently Angeletos (2007), He and Krishnamurty (2012) and Brunnermeier and Sannikov (2014). It features incomplete financial markets and two types of risk-averse agents: households and entrepreneurs subject to idiosyncratic productivity shocks. Entrepreneurs want to finance their investments by raising outside funds, but face the problem that their revenues are private information and standard outside equity finance is therefore impossible. Putting this corporate finance perspective at center stage allows us to address the classic macroeconomic problems outlined above from a new angle and to provide surprisingly precise answers and solutions in closed form.

A benevolent social planner in our economy faces the task of redistributing unobservable output among entrepreneurs and households such as to share idiosyncratic production risk and optimize intertemporal production and consumption. Intuitively, she must design a dynamic multi-agent mechanism that rewards entrepreneurs with high output for sharing some of this output with low performing entrepreneurs and households, without leaving them so much surplus as to jeopardize the risk-sharing objective. We have solved this mechanism-design problem in our more technical companion paper Biais et al. (2024). In this paper, we focus on implementation and show that the resulting constrained optimal allocations can be decentralized if entrepreneurs can issue short-term debt, by giving the government two simple fiscal instruments: the issuance of liquid public debt and continuous linear taxation or subsidization of wealth of entrepreneurs and households at different rates.

In fact, if their debt is very short-term, entrepreneurs can react flexibly to output shocks by issuing varying amounts of debt, even when the basic information asymmetry between entrepreneurs and outsiders prevents them from issuing state-contingent claims. We assume that such instantaneous debt transactions are frictionless. Furthermore, entrepreneurs can use inside equity as a further margin of adjustment, and we show that they optimally use both margins simultaneously in order to downsize the balance sheet after negative production shocks and size up after positive shocks. Short-term debt therefore is a "*flexible hard claim*" that provides second-best managerial incentives in the spirit of Grossman and Hart (1983)'s theory of debt as an incentive mechanism. It is flexible because it can be scaled continuously and without frictions, and it is hard because, knowing that a firm can adjust its leverage flexibly, outsiders can enforce the remaining claims perfectly. In fact, since idiosyncratic production shocks in our model are assumed to follow a Brownian motion, debt is safe, as in much of the classic asset pricing literature following Merton (1971).

The reason why the two fiscal instruments, together with initial one-shot redistribution, achieve optimal decentralization is that the entrepreneurs' scaling decisions in response to their production shocks follow the same logic as the social planner's redistribution policy: entrepreneurs with higher instantaneous output can scale up (by issuing debt that replaces that of less successful ones), while entrepreneurs with unfavorable production shocks must scale down to avoid bankruptcy (which they will, because bankruptcy is less attractive than continuation even at very small size). Since the entrepreneurs' production decisions depend on their net worth, there is a role for public debt in affecting firms' balance sheets in equilibrium. To this end, the government optimally mimics private debt and issues safe short-term debt, too, which is a perfect substitute to private debt. Since this intervention benefits agents asymmetrically, it must be accompanied by redistributive taxation, which depends on the weights in the welfare function.

As we discuss in Section 5, the decentralization result is akin to the Second Welfare Theorem. Just as in classic complete-market settings à la Arrow-Debreu, the public planner can redistribute resources through appropriate taxes or transfers and affect production decisions. In the equilibrium of the decentralized economy, fiscal policy affects the aggregate balance sheet of the private sector, and in particular private leverage, through public bonds and taxes. Issuing public debt and distributing the proceeds to entrepreneurs and households has three effects: a balance sheet effect, an interest rate effect, and a growth effect. The balance sheet effect reduces the net leverage of entrepreneurs and increases their incentives to undertake risky investments at any given risk-free interest rate. This directly benefits entrepreneurs and increases growth. But to clear the market for capital, the increased supply of bonds increases the risk-free interest rate. This partially counteracts the balance sheet effect for entrepreneurs, as they have to pay higher interest on their lower net debt, and it directly benefits households. Finally, issuing public debt increases the aggregate wealth of the private sector, which stimulates aggregate consumption, which in turn has a negative impact on output growth. Despite these counterveiling effects, the optimal level of government debt is always positive and balances these different effects, depending on the weight of entrepreneurs and households in the social welfare function. In particular, Ricardian equivalence does not hold, because changing the firms' budget constraints has real effects.[1]

The Pareto optima in our model involve redistribution and therefore are not always Pareto improvements over the Laisser-Faire allocation of no government intervention. However, once we control for redistribution and start out with an allocation in which taxes and transfers are used optimally, but the government budget is balanced, we obtain the remarkable result that optimal public debt issuance increases the expected utility of households and entrepreneurs at every point in time and for all realizations of individual productivity shocks. Public debt issuance thus constitutes what Aguiar et al. (2024) call a "Robust Pareto Improvement" over a situation in which only taxes are used to finance government expenditures and redistribute income. In fact, government debt acts like a rising tide lifting all boats, by reducing the entrepreneurs' exposure to household lending and thus their leverage, which improves risky investment incentives. This translates into universal Pareto improvements because the government can use taxes and transfers to re-distribute some of resulting improvements.

However, as argued forcefully by Atkeson and Lucas (1992), Pareto optimality can be in stark conflict with equality. They show in a simple dynamic infinite-horizon exchange economy with private information, that it is Pareto optimal to let almost all agents' utility converge to their minimum levels for $t \to \infty$. The reason is as in the classic partial-equilibrium analysis of Thomas and Worrall (1990) that it is cheaper to reward risk-averse agents for truth-telling later rather than earlier, so that the incentive-optimal consumption profiles are backloaded and favor the lucky very few very strongly over the poor vast majority. Our endogenous growth model adds a new dimension to this argument because correct incentives stimulate growth. Indeed, as our analysis in Section 5 shows, the famous Thomas-Worrall-Atkeson-Lucas immiseration result holds if average productivity in our economy is

[1]This happens although many of the usual assumptions for Ricardian equivalence hold: agents are fully rational and forward looking, everybody can borrow and lend at the same safe interest rate, and the path of government expenditures is fixed. But as Barro (1974) himself points out in his classical paper, a further assumption needed is that "the marginal net-wealth effect of government bonds is close to zero." The whole point of the optimal policy considered here is to violate this assumption.

sufficiently low, but it fails to hold if productivity is larger. In that case, in the unique welfare optimal equilibrium of our model the economy grows at a positive steady state rate. Although inequality (measured by the coefficient of variation of the wealth of productive agents) increases without bounds for $t \to \infty$, almost every agent is eventually strictly better off than at time $t = 0$. Hence, immiseration is an interesting, but special phenomenon that does not necessarily arise in productive economies.

Equipped with our version of the Second Welfare Theorem, we can address the central questions asked in the first paragraph above. In particular, questions about the optimal public debt-to-GDP ratio or the role of optimal taxes versus public debt can be answered simply and explicitly. Interestingly, changing the weight of the productive sector (i.e., the welfare weight of entrepreneurs) in the social welfare function fundamentally changes the optimal structure of public finances. As we discuss in Section 6, the impact of welfare weights on equilibrium outcomes such as the debt-to-GDP ratio or the growth rate is non-monotonic, as the tradeoffs between the three major effects of fiscal policy discussed above change.

In terms of fiscal policy, we can use this model to revisit the classic question of how to fund public expenditure between debt and taxes. In the model, public expenditure is exogenous and non-discretionary, i.e. needed to provide fixed public infrastructure and public goods. In this framework, in any welfare-optimal equilibrium the debt-to-GDP ratio is independent of public spending, while taxes depend on it. Public debt is a matter of structural policy towards incomplete financial markets and growth, and taxes provide the residual required for funding public expenditure. Given that we do not consider aggregate risk, we cannot address optimal aggregate fiscal policy smoothing, but can only do comparative statics, which is equivalent to considering unanticipated public expenditure shocks. However, even this limited approach yields an interesting dynamic interpretation: fiscal policy should be such that higher public spending needs are covered by increased taxes, but do not affect the debt-to-GDP ratio. While this result certainly depends on the structure of our simple endogenous growth model, it seems that it contains a more general message for the current political debate.

Our work builds on and contributes to different strands of the macroeconomic literature on fiscal policy with agent heterogeneity, which we review in the next section. However, it is worth emphasizing one issue of more than recent interest. As our model yields explicit analytic expressions for the equilibrium growth and interest rates, the analysis can directly address the question of the structural determinants of the difference $r - g$.[2] As noted above, we find that the interest rate

[2]Some of the important recent contributions to the $r - g$ debate are Barro (2023), Blanchard (2019), Brunnermeier et al. (2021), Cochrane (2019), Cochrane (2022), Dumas et al. (2022), Reis (2021).

may exceed the growth rate of GDP in some parameter constellations, in particular when entrepreneurs' weight in social welfare is large, and the opposite occurs when it is low.[3] In the first case ($g < r$), the value of outstanding debt at each date is equal to the net present value of all future primary surpluses, in line with classic textbooks. In the second case, $r < g$ and there is a permanent and growing primary deficit. In this case, perhaps surprisingly, the public debt-to-GDP ratio is small, and the government optimally runs a Ponzi scheme: it eternally repays old debt by issuing new one.

The remainder of this paper is organized as follows. In the next section, we provide a more detailed discussion of the literature. The basic model in terms of preferences, technology, and endowments is set out in Section 3, where we also characterize (constrained) optimal allocations by drawing on our work in Biais et al. (2024). In Section 4, we introduce a market environment, characterize individually optimal decisions, and show what policy instruments are needed to decentralize the Pareto optimal allocation. The general equilibrium analysis is presented in Section 5, which also discusses some key distributional problems arising in equilibrium and Robust Pareto Improvements in the sense of Aguiar et al. (2024). In Section 6, we discuss the roles of debt and taxes under optimal fiscal policy. The implications of fiscal policy for optimal growth, interest rates, and the government budget are explored in Section 7. The conclusion in Section 8 presents a brief outlook and some generalizations in further research. A few formal proofs are relegated to the Appendix, and a detailed proof of individual optimization outcomes is added as an online appendix.

2 Relation to the literature

Our paper is related to several strands of the literature. First, since the early overlapping generations models dating back to Diamond (1965), a sizable literature has examined how fiscal policy influences the relations between the real interest rate (r), the growth rate of output (g), and the marginal product of capital (μ).[4] A recent strand of this literature re-examines this question in settings with infinitely-lived agents, using continuous-time methods from asset pricing. Building on the seminal contributions by He and Krishnamurty (2012), Brunnermeier and Sannikov (2014), and Di Tella (2017), this literature considers economies with aggregate risk

[3]The full comparative statics of the determinants of this tradeoff is given in Proposition 11.

[4]In the context of the overlapping generations framework, a recent debate about the sustainability of fiscal policy has focused on when and why governments can use Ponzi Schemes – eternal deficits without relying on taxation when $r < g$ – in the presence of uncertain production returns (Blanchard and Weil (2001), Blanchard (2019), Jiang et al. (2024) Dumas et al. (2022)). Dumas et al. (2022) endogenize the structural deficit in the form of an underfunded social security scheme and characterizes debt capacity limits.

and studies the emergence and amplification of financial crises, as well as the role of intermediaries in this dynamic. Our work is complementary, as we focus on optimal fiscal policy in the long run when financial markets are limited by informational frictions between firms and outside investors and there are conflicting interests between households and firms. Unlike Basak and Cuoco (1998) and Brunnermeier and Sannikov (2014, 2016), we follow the traditional approach in corporate finance and assume that production, rather than capital accumulation, is risky (both approaches are in general largely equivalent). This is because this approach lends itself better to the information-based view of financial frictions, which is at the heart of our model.

Regarding the welfare-enhancing role of public debt, our work builds on Brunnermeier et al. (2021), who focus on how to integrate a bubble term representing government expenditures – without ever raising taxes for them – into the fiscal theory of the price level in the presence of idiosyncratic risks and incomplete markets. They determine what they call the optimal "bubble mining rate", which is the optimal rate of issuing government debt. Brunnermeier et al. (2024) extend this approach and resolve the "public debt valuation puzzle", by showing that the price of debt is procyclical, since the bubble term rises in bad times.

There is not much work in macro-finance rationalizing market incompleteness along our lines.[5] An early macroeconomic classic on the question why financial markets in an economy with private information are incomplete and how this affects welfare is Cole and Kocherlakota (2001) who study an endowment economy, in which agents can save privately by storing part of their endowments and where storage *and* endowment size are private information. If the return on saving is sufficiently high, they show that a Pareto optimum exists that can be decentralized as the equilibrium outcome of a financial market with short-term debt. Since their model has a finite horizon and no production risk, they cannot speak to most of the issues that are central to our paper: interest and growth, long-term inequality and immiseration, underinvestment by firms, and how this is alleviated by government debt and taxes. One important recent contribution that models entrepreneurial moral hazard explicitly is Di Tella (2020), which builds on Di Tella and Sannikov (2021). The main differences are that Di Tella (2020) treats money and thus has money in the utility function, that his model has hidden savings and hidden trading, that agents can privately write long-term contracts with full commitment, and that the agency problem solved in Di Tella and Sannikov (2021) is a one-agent

[5]The typical approach is to simply *assume* an incomplete market structure, as in Brunnermeier and Sannikov (2014): "A constraint on expert equity issuance *can* be justified in many ways, e.g., through the existence of an agency problem between the experts and households."... "For now, we focus on the simplest *assumption* that delivers the main results of this article: experts must retain 100 percent of their equity and can issue only risk-free debt."(emphasis added)

problem, whereas the agency problem in Biais et al. (2024) that underlies our analysis explicitly features many agents and considers the whole wealth distribution as a state variable.

The contractual approach to macroeconomic foundations is probably more prominent in the literature on contracting under limited commitment (see Golosov et al. (2016) for an authoritative survey and Krueger and Uhlig (2022) for a full analysis of decentralizing partial insurance outcomes in general equilibrium under limited commitment with log utility). This seems to be largely due to the immiseration problem,[6] which is less of a concern in our endogenous-growth model: if the social contract promises optimal growth for (almost) everybody through incentives, debt, and taxes, then there is no need to revoke it, even if inequality becomes large.

There is a broad strand of early macroeconomic literature that takes borrowing constraints or similar restrictions as exogenous, including Bewley (1983), Imrohoroğlu (1989), Huggett (1993) and in particular Aiyagari (1994) on self-insurance by households against income fluctuations. This literature is large and was surveyed by Heathcote et al. (2009) and Krueger et al. (2016). As discussed above, we are complementary to this approach and take it one step further, by following Angeletos (2007), who has enriched the neoclassical growth model by uninsured idiosyncratic investment risk and characterized the macroeconomic effects of this feature. Our model in many aspects is a generalization of his.

The new heterogeneous-agents macro models with incomplete markets are a natural framework to analyze fiscal policy. An emerging literature has indeed made important progress with such models and their calibrations, which often have a rich institutional structure (see, in particular, Le Grand and Ragot (2023), Dyrda and Pedroni (2023), and Greulich et al. (2023)). We are complementary to this work by developing a tractable growth model with heterogeneous agents, in which we can derive optimal fiscal policy from first principles, based on a version of the Second Welfare Theorem. The model features a finance-growth channel that is different from, in particular, Aguiar et al. (2024) who develop the concept of Robust Pareto Optimality in the framework of a generalized Aiyagari-Huggett model. Their generality comes at the expense of some restrictions that we can avoid in our theory, such as the assumptions that the form of tax schedules (linear) and transfers (lump-sum) is exogenously given, that equilibria are stationary, that there are exogenous borrowing constraints, or that $r < g$.

A classic part of the literature relevant for our work examines the role of government debt as an asset that can help overcome financial frictions. In a seminal paper, Woodford (1990) demonstrates that public debt can enhance economic effi-

[6]And its prominence in macroeconomics post Atkeson and Lucas (1992): Immiseration is "often regarded as being the hallmark result of dynamic social contracting in the presence of private information" (Kocherlakota (2010), p. 70).

ciency when imperfect financial intermediation prevents some agents from borrowing against their future income at the interest rates available to the government. By issuing debt, the government supplies liquidity-constrained agents with a highly liquid asset in exchange for future tax liabilities. This shift increases the share of liquid assets in private sector wealth, thereby improving the ability of liquidity-constrained agents to smooth consumption in response to income fluctuations and spending opportunities. In a similar vein, Aiyagari and McGrattan (1998) develop a model in which households face a borrowing constraint, which generates a precautionary savings motive. Government debt loosens borrowing constraints and enhances the liquidity of households, which improves consumption smoothing. These papers also stress the cost of higher government debt via incentive effects and crowding-out effects on investment.[7]

In our paper, there are neither borrowing nor liquidity constraints, firms can borrow freely as long as this is incentive-compatible. Public debt mitigates these problems and gives rise to a tradeoff between underinvestment of firms and higher interest rates, that can be interpreted and solved as a multi-agent mechanism design problem. Closer to our subject, and as in the present paper, Holmström and Tirole (1998) take a corporate finance perspective, which they use to study optimal contracts between firms and outside investors in the presence of managerial moral hazard. They show that the public supply of liquidity is not necessary in an economy with no aggregate uncertainty and in which intermediaries coordinate the use of scarce private liquidity. Our work challenges this conclusion and argues that the underlying distributional problems necessitate a government with the power to tax and re-distribute.

Last but not least, our work is naturally linked to the literature on continuous-time corporate finance. Building on Leland (1994), the contingent claims literature evaluating the tradeoff between bankruptcy costs and tax benefits of debt typically takes the structure of securities as given and assumes costless equity issuance and long-term debt with positive adjustment costs, exactly opposite to our assumptions in the present paper. As pointed out, e.g., by Abel (2018) and Bolton et al. (2021), this structure is difficult to reconcile with some of the empirical evidence on leverage dynamics, which is one reason why these authors model capital structure with instantaneous, frictionless debt, as in the Merton-based approach of our paper (Merton, 1971). Different from all these papers we take a strict agency view, which rules out outside equity as a source of funding. This is in the tradition of the seminal work by DeMarzo and Fishman (2007), DeMarzo and Sannikov (2006), and Biais et al. (2007) that use recursive methods to characterize the optimal security

[7] Similarly, Angeletos et al. (2023) explore how public debt can be used as collateral or a liquidity buffer in order to ease financial frictions. Since public debt lowers the liquidity premium but increases the cost of borrowing for the government, there exists a long-run optimal level of public debt.

structure for funding single-firm investment projects under asymmetric information. These papers consider one-shot investment projects; in this perspective, our work is close to Biais et al. (2010) who were the first to study the problem of repeated investments in such models. Since we rule out the possibility of pledging project output to outsiders, in terms of capital structure our theory implies a combination of inside equity and riskless instantaneous debt with maximum managerial flexibility in the face of earnings risk. As discussed above, this leads to the concept of short-term debt as a flexible hard claim and links our result on the implementation of the optimal funding mechanism of entrepreneurs to the theory of debt as an optimal incentive device in the spirit of Grossman and Hart (1983).

3 The Model

3.1 The Macroeconomic Set-Up

The economy features two classes of agents, a mass 1 continuum of entrepreneurs and a mass 1 continuum of households, plus a government. Time is continuous: $t \in [0, \infty)$. There is only one physical good that can be consumed or invested, but cannot be stored privately for future consumption.[8]

Entrepreneurs run their own firms that are risky in the sense that they produce random output at each point in time. These random outputs are not publicly observable, which implies that entrepreneurs cannot fully share their risks.[9] The physical good is initially held by households and entrepreneurs, but only entrepreneurs can invest in productive technologies. Households cannot. For simplicity we also assume that households do not work and that they live off their savings. They are identical, and are not subject to individual shocks. Without loss of generality, we can therefore aggregate them into a single representative household (the "household sector"). We cannot do this for the productive sector. Even if starting out identically, entrepreneurs become heterogeneous after $t = 0$ because of their idiosyncratic productivity shocks: lucky entrepreneurs grow bigger while unlucky ones operate less capital.

[8]Since we assume that invested goods and consumption goods are identical, investment also serves as a form of publicly observable storage (where the amount "stored" is observable, but not its result). Ruling out additional privately observable storage simplifies the conclusion in Section 4 below that the efficient allocation can be decentralized by riskless short-term debt. The classic analysis by Cole and Kocherlakota (2001) shows that this also holds when agents alternatively have access to safe hidden storage (where the amount stored is unobservable, but the result is riskless and sufficiently high). Di Tella (2020) shows that this also holds in an environment with money in which agents have private bank accounts.

[9]This is as in Basak and Cuoco (1998) and much of the subsequent literature, the difference being that in our work the trading restriction is explicitly derived from an underlying informational friction, whereas it is exogenous in the cited literature. Brunnermeier and Sannikov (2016) and Di Tella (2020) generalize the existing literature by assuming that firms can sell equity, but must hold an exogenous minimum fraction of it.

The government must finance an exogenous stream of public expenditures and can redistribute wealth between households and entrepreneurs, as long as it respects the informational constraints described above.

The economy considered here is very similar to that in Biais et al. (2024), where we characterize constrained efficient allocations in a dynamic contracting model between one principal and many agents. The two main differences with respect to that model are, first, that here we have a government with a fixed level of public expenditures instead of a self-interested principal, and second, that we add a household sector that consumes but does not produce. In Biais et al. (2024), we do not consider decentralization and markets, which is the focus of the present paper.

3.2 Technology, Preferences, and Endowments

Entrepreneurs are indexed by $i \in [0,1]$, with random initial endowments $\tilde{e}^i > 0$. We denote the entrepreneurs' aggregate initial endowment by $E = \mathbb{E}\tilde{e}^i$. The representative household has initial endowment H.

Entrepreneurs all have the same technology and produce random output at each point in time: if k_t^i is the volume of capital managed by entrepreneur i at date t, his instantaneous output net of depreciation is

$$dy_t^i = k_t^i\big[\mu dt + \sigma dz_t^i\big] \tag{1}$$

where $\mu > 0$ is the average instantaneous return net of depreciation, $\sigma \geq 0$ is the volatility of instantaneous returns, and the (z_t^i) are firm-specific i.i.d. Brownian motions, whose increments can be interpreted as idiosyncratic productivity shocks. Let $(\mathcal{F}_t)_{t\geq 0}$ be the augmented filtration generated by the (z_t^i). All processes associated with entrepreneurs and households are assumed to be square-integrable and adapted to $(\mathcal{F}_t)$. Importantly, the instantaneous output dy_t^i can only be observed by the entrepreneur.

Since the idiosyncratic production shocks of entrepreneurs are independent, aggregate capital can be considered as deterministic due to the law of large numbers.[10] Hence, if

$$K_t = \int_0^1 k_t^i di \tag{2}$$

denotes aggregate capital at time t, aggregate production (gross domestic product) at time t is

$$dY_t = \mu K_t dt \tag{3}$$

At each time t, the government must finance public goods that cost γK_t, where γ

[10] The analysis can be generalized to aggregate shocks. See the Conclusion for a brief discussion.

is an exogenous fraction of the aggregate capital stock.[11] Government expenditures as a share of GDP at date t are thus an exogenous constant γ/μ.[12]

Entrepreneurs (E) and households (H) have identical preferences over consumption streams $(c_t^i)_{t=0}^{\infty}$ and $(c_t^H)_{t=0}^{\infty}$. At each time t, they maximize

$$\mathbb{E}_t \int_t^{\infty} e^{-\rho(s-t)} \log\left(c_s^k\right) ds, \quad k = i, H,$$

where the expectations operator for agent i refers to the probability measure of the (z_t^i) and the associated filtration $(\mathcal{F}_t)$. $\rho > 0$ is the common discount rate.[13]

3.3 Constrained Optimal Allocations

By definition, an allocation consists of a set of consumption paths (c_t^i) for entrepreneurs $i \in [0, 1]$ and (c_t^H) for the representative household, together with capital allocations (k_t^i) for entrepreneurs. Note that all these paths are a priori random. Since their output is only privately observable, entrepreneurs can divert part of this output and consume it secretly, which interferes with aggregate risk-sharing. To control this diversion, a social planner must impose incentive-compatibility conditions. These conditions balance an entrepreneur's private gain from secret consumption (the assumption being that the good must be either invested or consumed) against her benefit from being able to operate more capital, which results in higher future returns.

We assume that the planner's welfare objective is a weighted average of entrepreneurs' and households' expected utilities. We further assume that the planner puts equal weight on all entrepreneurs, regardless of their identity or initial equity endowment. Hence, she decides before individual initial equity positions $\tilde{e}^i$ are known, under the Rawlsian "veil of ignorance".[14] Denote this common expected value by V^E and that of the household sector by V^H. The social planner therefore maximizes

$$W = \eta V^E + (1 - \eta) V^H \tag{4}$$

[11]We do not model the social utility generated by these expenditures explicitly and, therefore, say nothing about their optimal level.

[12]The assumption that government expenditure is a fixed fraction of GDP simplifies the analysis. Without linearity, the simple additive structure of (7) below is lost, and explicit calculations become more difficult.

[13]With logarithmic utility calculations and some results become particularly simple. However, as we have examined in ongoing work, the equilibrium analysis of the present paper can be done with more general CRRA preferences.

[14]An alternative assumption would be that the social planner must respect the initial inequality of the distribution of endowments $\tilde{e}^i$. We discuss the problem of inequality and its dynamics in Section 6.

where $\eta \in (0,1)$ is the welfare weight of entrepreneurs and

$$V^H = \mathbb{E}\int_0^\infty e^{-\rho t} u(c_t^H) dt \quad (5)$$

$$V^E = \mathbb{E}\int_0^\infty e^{-\rho t} u(c_t^i) dt \quad (6)$$

Here, $\mathbb{E}$ denotes the expectation at time 0 over the paths (c_t^i), (k_t^i), $i \in [0,1]$, and (c_t^H), which must be chosen such that the right-hand side of (6) does not depend on i. Maximization is subject to a number of constraints. Clearly, for efficiency reasons, all physical initial resources must be given to entrepreneurs. Hence, initial aggregate capital is

$$K = H + E.$$

Furthermore,

- by assumption, all entrepreneurs have the same expected utility in $t = 0$,
- all entrepreneurs' consumption and investment decisions must be incentive-compatible,
- aggregate capital (2) evolves according to

$$\dot{K}_t = (\mu - \gamma)K_t - c_t^H - \int_0^1 c_t^i di. \quad (7)$$

Here, (7) is a standard IS equation: all available output is used either for private consumption, public spending or invested. As noted above, in Biais et al. (2024), we solve an optimization problem similar to this as a multi-agent dynamic mechanism-design problem by means of mean-field control theory.[15] For log utility, this approach implies that the solution to our problem has the following properties:

- capital is initially allocated equally between entrepreneurs: $k_0^i = K_0 = K$ for all i
- There are numbers $g > 0$ and $x > 0$ such that capital is continually reallocated among entrepreneurs as a function of performance as follows:

$$\frac{dk_t^i}{k_t^i} = g dt + \sigma x dz_t^i \quad (8)$$

- entrepreneurs consume an amount of output equal to the fraction ρ/x of the capital they manage:

$$c_t^i = \frac{\rho}{x} k_t^i \quad (9)$$

[15] The analysis in Biais et al. (2024) is technically involved. It transforms the agents' incentive problem by generalizing Sannikov (2008)'s state-space approach to many agents. For a continuum, this makes the distribution of individual continuation utilities one state variable of the optimization problem, which requires developing and solving a Hamilton-Jacobi-Bellman equation in an appropriate infinite-dimensional space. See Achdou et al. (2014) for a pioneering contribution with respect to HJB theory in macroeconomics and Carmona and Delarue (2018) for a mathematical introduction.

- households consume a constant fraction χ of aggregate capital K_t:

$$c_t^H = \chi K_t. \tag{10}$$

Together with the above properties, equation (7) immediately implies that the growth rate of aggregate capital in (8) is deterministic and satisfies

$$g = \mu - \gamma - \chi - \frac{\rho}{x}. \tag{11}$$

3.4 Welfare Optima

Using the above characterization, we can compute the welfare optimal expected utility of the representative household as

$$\begin{aligned} \rho V^H &= \rho \mathbb{E} \int_0^\infty e^{-\rho t} \log(\chi K_t)\, dt \\ &= \log \chi + \log K_0 + \mathbb{E} \int_0^\infty e^{-\rho t} d \log K_t \\ &= \log \chi + \log K_0 + \frac{1}{\rho} g \end{aligned}$$

The expected utility of entrepreneurs at time 0 has a similar expression, except for the risk premium resulting from the individual risk exposure in (8). Of course, as noted above, after time 0, entrepreneurs will be heterogeneous. But from an ex ante perspective, since technologies are i.i.d. and the initial re-distribution of endowments equalizes the initial capital positions, they all have the same expected utility, given by[16]

$$\begin{aligned} \rho V^E &= \rho \mathbb{E} \int_0^\infty e^{-\rho t} \log\left(\frac{\rho k_t^i}{x}\right) dt \\ &= \log \frac{\rho}{x} + \log K_0 + \frac{1}{\rho}\Big(g - \frac{\sigma^2 x^2}{2}\Big). \end{aligned}$$

The social planner chooses x and χ so as to maximize the weighted average

$$\begin{aligned} W &= \eta V^E + (1-\eta) V^H \\ &= (1-\eta) \log \chi + \eta \log \frac{\rho}{x} + \log K + \frac{1}{\rho}\big(\mu - \gamma - \chi - \frac{\rho}{x}\big) - \eta \frac{\sigma^2 x^2}{2\rho} \end{aligned} \tag{12}$$

The optimal χ and x are uniquely determined by the first-order conditions

$$\chi = \rho(1-\eta) \tag{13}$$

$$\frac{\sigma^2}{\rho} x^3 + x - \frac{1}{\eta} = 0. \tag{14}$$

This leads to the following characterization of the welfare optimum.

[16]For completeness we provide a brief proof in Appendix A.1.

Proposition 1. *The unique constrained optimal allocation that maximizes welfare W satisfies conditions (8), (9), and (10), with the parameter values $x^* > 0$ and $\chi^* > 0$ given by (13)-(14). The optimal growth rate of aggregate capital is constant and equal to*

$$g^* = \mu - \gamma - \chi^* - \frac{\rho}{x^*} = \mu - \gamma - \rho - \frac{\rho\sigma^2 x^*}{\rho + \sigma^2 x^{*\,2}} \tag{15}$$

The optimal allocation is surprisingly simple. It is stationary, and consumption and investment is controlled by two positive numbers, χ and x, that are given by two explicit simple formulas. Growth is stationary, because there is no aggregate risk. The entrepreneurs' optimal performance sensitivity x^* decreases in the entrepreneurs' welfare weight η, which implies that, ceteris paribus, entrepreneurial consumption increases and growth decreases. The effect is counteracted, however, by lower household consumption. We will return to this discussion in Section 6 below.

4 Decentralized Decisions

4.1 The Market Environment

We now specify how the welfare maximizing allocation of Proposition 1 can be decentralized as the unique rational expectations equilibrium of a private property market economy with fiscal policy. In this sense, the incomplete market structure is endogenous. First, in this section, we determine the optimal individual decisions in such a market structure and the policies necessary to implement them. In the next section we study general equilibrium and its distributional properties.

As noted above, there can be no market for claims based on the entrepreneurs' output, as this is private information. Hence, entrepreneurs can only issue risk-free debt. In order to be able to react to their private output shocks, they must be able to adjust their leverage flexibly. As in Cole and Kocherlakota (2001) and other papers, we therefore assume that such instantaneous debt markets exist and are frictionless.[17] Let r_t be the (instantaneous) interest on this debt. Hence, at each moment t, one unit of the good (or capital) can be exchanged costlessly against one unit of safe debt, which instantaneously (i.e., at time $t + dt$) pays r_t.[18]

The government influences economic activity by using three fiscal instruments, which are consistent with its informational constraints (balance sheets are observable, but not per-period cash flows). First, it can at all times issue safe short-term public debt B_t, which is a perfect substitute for private debt. Second, it can make

[17] In fact, since debt expires instantaneously, we only need to assume that entrepreneurs can issue debt instantaneously. A secondary market is not needed.

[18] Unlike in Cole and Kocherlakota (2001) who work with a finite model, investment returns in our model can become negative, so bankruptcy is an issue. In Section 4.2 below we verify that under the optimal investment strategy private debt is indeed safe.

initial transfers to entrepreneurs and households, either in the form of goods or public debt, whose aggregate values we denote by L^E and L^H, respectively.[19] And third, it can continuously tax the wealth of entrepreneurs and households by means of linear taxes, at rates τ_t^E and τ_t^H, respectively. Initial transfers and instantaneous taxes can all be negative, thus allowing for taxes and transfers at all stages. Note that these instruments are by no means general. As we shall see, however, they suffice to implement the second-best.

At time $t = 0$, after the government's initial intervention, net wealth is H_0 for the representative household, e_0^i for entrepreneur i, and aggregate wealth (equity) in the productive sector is

$$E_0 = \int_0^1 e_0^i di.$$

The aggregate wealth of the private sector is

$$E_0 + H_0 = K_0 + B_0$$

where $K_0 = K$ is the initial stock of physical capital. Thus the government can modify the balance sheet of the private sector by shifting goods from households to entrepreneurs and by issuing and distributing public debt, which is a paper claim on the good. However, the government cannot produce any output, so that the aggregate capital stock of the economy is still K_0.

Government debt evolves according to

$$\dot{B}_t = \gamma K_t + r_t B_t - T_t, \tag{16}$$

where T_t is net aggregate tax revenue (tax revenue minus subsidies) at time $t > 0$.

4.2 Individual Decisions

The individual decision problems of households and entrepreneurs in this economy are standard and yield well-known solutions going back to Merton (1971).[20]

After the initial government intervention, the representative household has initial net worth $H_0 > 0$ at time $t = 0$, no further income later, and saves via private and public bonds, which are perfect substitutes. There is no other form of savings, since the good cannot be stored. Hence, the household chooses a consumption path $c^H = \left(c_t^H\right)_{t \geq 0}$ that solves

$$\max_{c^H} \int_0^\infty e^{-\rho t} \log c_t^H dt,$$

[19] Since individual initial endowments of entrepreneurs are heterogeneous, entrepreneur i receives $L^i = e_0^i - \bar{e}^i = e_0^i - E_0$. This initial redistribution among entrepreneurs reflects our assumption that the social optimum in (12) has one single utility target V^E for all entrepreneurs.

[20] For completeness, we present these solutions in detail in the online Appendix A.3.

subject to the equation of motion of wealth

$$\dot{H}_t = (r_t - \tau_t^H)H_t - c_t^H. \tag{17}$$

This is a standard dynamic programming problem. The solution is

$$c_t^H = \rho H_t \tag{18}$$

for all $t \in [0, \infty)$.

The entrepreneurs' problem under decentralization is a bit more complex. At time t, entrepreneur i has wealth e_t^i, chooses productive capital k_t^i, and adjusts her debt level d_t^i such as to satisfy her balance sheet constraint

$$k_t^i = d_t^i + e_t^i. \tag{19}$$

We allow debt d_t^i to be negative, in which case the entrepreneur has no debt but invests in bonds issued by other entrepreneurs or the government. The entrepreneur's flow of funds is given by

$$k_t^i[\mu dt + \sigma dz_t^i] = [r_t d_t^i + \tau_t^E e_t^i + c_t^i]dt + de_t^i, \tag{20}$$

where the left-hand side represents earnings before interest and taxes and the right-hand side is the sum of interest payments, taxes, private consumption (dividends), and retained earnings as a residual. (20) thus reflects the simple accounting identity:

$$\text{EBIT} = \text{interest} + \text{taxes} + \text{dividends} + \text{retained earnings}.$$

The entrepreneur chooses a path of k_t^i, d_t^i, c_t^i, $t \geq 0$ that solves

$$\max_{k^i, d^i, c^i} \mathbb{E} \int_0^\infty e^{-\rho t} \log c_t^i dt$$

subject to the balance sheet constraint (19) and the law of motion (20) for each $t \geq 0$. If $r_t < \mu$ (which we verify later), the Bellman Equation yields the standard solution

$$c_t^i = \rho e_t^i \tag{21}$$

$$k_t^i = \frac{\mu - r_t}{\sigma^2} e_t^i \tag{22}$$

and the stochastic law of motion for entrepreneurial equity

$$de_t^i = \left[\left(\frac{\mu - r_t}{\sigma} \right)^2 + r_t - \tau_t^E - \rho \right] e_t^i dt + \frac{\mu - r_t}{\sigma} e_t^i dz_t^i. \tag{23}$$

Note that the flow of funds equation (20) assumes that the entrepreneur is always able and willing to pay the interest on her debt. If she decided to default, her assets can be seized, entailing zero consumption and minus infinite utility ever after. Hence, if this reaction happens with some probability, strategic default is not an issue.[21] Moreover, the stochastic differential equation (23) describes a Geometric Brownian Motion:[22]

$$e_t^i = e_0^i exp\left(\int_0^t \left[r_s - \tau_s^E - \rho + \frac{1}{2}\left(\frac{\mu - r_s}{\sigma}\right)^2\right] ds + \int_0^t \frac{\mu - r_s}{\sigma} dz_s^i\right) \tag{24}$$

Therefore, equity is always positive for all entrepreneurs: involuntary default thus is no issue either. The capital k_t^i of entrepreneur i evolves as a function of her performance, but she never defaults in equilibrium. In fact, condition (22) implies that the capital-to-equity ratio

$$x_t = \frac{k_t^i}{e_t^i} \tag{25}$$

is identical across entrepreneurs. Using this same leverage target, all entrepreneurs adjust their debt and capital continuously in response to their earnings shocks. After a positive shock, they increase capital and issue more debt; after a negative shock, they do the opposite.[23] Hence, Brownian productivity shocks are not enough to drive entrepreneurs into bankruptcy.[24]

4.3 Optimal Policy

Suppose a social planner wants to decentralize the optimal allocation derived in Section 3, and consider any pair (χ, x) as given in (8), (9) and (10).[25]

By (10) and (18), decentralizing household consumption requires

$$\chi K_t = \rho H_t \tag{26}$$

for all t. In particular, initial household wealth must be

$$H_0 = \frac{\chi}{\rho} K_0. \tag{27}$$

Furthermore, τ_t^H must be chosen such that H_t grows at the same rate as K_t, namely g. By (11), (17), and (18) this requires that

$$\tau_t^H = r_t - g - \rho. \tag{28}$$

[21]There is a large literature on strategic default, which we do not need to discuss here. See Hart and Moore (1998) or Bolton and Scharfstein (1996) for foundational work and Fan and Sundaresan (2000) for an early classic in continuous time.

[22]See, e.g., Shreve (2004), p. 147-8.

[23]This is different from the logic of the monetary model of Di Tella (2020), where idiosyncratic shocks affect the agents' capital stocks and the safe asset (money) is held for precautionary saving. There, entrepreneurs with bad shocks use money to buy capital from entrepreneurs with good shocks.

[24]This is why Abel (2018) assumes discrete earnings shocks and Bolton et al. (2021) a jump-diffusion process.

[25]The decentralization argument in this subsection holds for any such pair, not just the optimal one defined in (13) and (14).

Decentralizing the constrained optimal decisions for entrepreneurs and implementing (8) and (9) by means of (21) and (22) requires setting

$$\frac{k_t^i}{e_t^i} = \frac{\mu - r_t}{\sigma^2} = x, \tag{29}$$

which implies that the interest rate must be constant:

$$r_t = r \equiv \mu - \sigma^2 x \tag{30}$$

Moreover, aggregate entrepreneurial equity must grow at the same rate as aggregate capital K_t. Hence, by (23) the tax rate on entrepreneurs' equity must satisfy

$$\tau_t^E = r_t - g - \rho + \left(\frac{\mu - r_t}{\sigma}\right)^2 = \tau_t^H + \sigma^2 x^2. \tag{31}$$

Finally, initial equity must be equalized across entrepreneurs and make (9) and (21) consistent, which implies

$$e_0^i = E_0 = \frac{K_0}{x}. \tag{32}$$

Using (14) to eliminate η, (13), (15), (30), and rearranging, we can summarise the implementation conditions (27), (28), (31), and (32) as the partial equilibrium result in the following proposition.

Proposition 2. *For the constrained optimal allocation characterized in subsection 3.4 to be the outcome of decentralized individual choices, it is necessary to set*

$$\begin{aligned} \tau_t^H &= \tau^H = \gamma - \frac{\sigma^4 x^3}{\rho + \sigma^2 x^2} && (33) \\ \tau_t^E &= \tau^E = \gamma - \frac{\sigma^4 x^3}{\rho + \sigma^2 x^2} + \sigma^2 x^2 && (34) \\ L^H &= H_0 - H = (1 - \eta)E - \eta H && (35) \\ L^E &= E_0 - E = \frac{1}{x}\left(H - (x - 1)E\right) && (36) \end{aligned}$$

where $x = x^*$ as given by (14). If $r_t = r^* \equiv \mu - \sigma^2 x^*$, (33)-(36) are also sufficient.

The implementation of the optimal incentive mechanism of Section 3.3 through short-term debt in (29) and Proposition 2 is remarkable from a corporate finance perspective. Short-term debt has two different roles in this optimality result. First, the variable x controls the entrepreneurs' incentives for investment and payout in (8) and (9). And second, the leverage ratio x provides a hard, but flexible individual claim that implements these incentives through the entrepreneurs' balance sheet. Importantly, short-term debt is a hard claim in the sense of, e.g., Hart and Moore (1995), as each entrepreneur i must and will pay out $r_t d_t^i$ to outsiders at all times. On the other hand, the claim is highly flexible as it reacts instantly to the entrepreneur's liquidity dy_t^i. Hence, the variable x controls incentives as well as payout obligations, in the spirit of Grossman and Hart (1983)'s theory of debt as an incentive mechanism.

5 Macroeconomic Equilibrium

This section determines the equilibria of the market economy defined in Section 4 when government taxes and transfers are given by (33)-(36). Let (r_t) be a trajectory of interest rates.

5.1 The Aggregate Balance Sheet

As noted in Section 4.2, households' aggregate wealth H_t is deterministic and optimally follows the law of motion

$$\dot{H}_t = \left(r_t - \tau^H - \rho\right) H_t. \tag{37}$$

This wealth is entirely invested in risk-free debt, and the household is indifferent between public debt and corporate debt. Let D_t^H and B_t^H denote the households' holdings of private and public debt, respectively. The households' balance sheet then is

$$H_t = D_t^H + B_t^H. \tag{38}$$

Optimal individual balance sheets of entrepreneurs follow random trajectories but thanks to the Law of Large Numbers, the aggregate balance sheet of the productive sector is deterministic. Denoting by B_t^E the entrepreneurs' aggregate holdings of public debt (which may be negative), it is given by:

$$\begin{array}{c|c} \text{Assets} & \text{Liabilities} \\ \hline K_t & D_t^H \\ B_t^E & E_t \end{array} \tag{39}$$

where E_t is total wealth of entrepreneurs, with dynamics implied by (23):

$$\dot{E}_t = \left[\left(\frac{\mu - r_t}{\sigma}\right)^2 + r_t - \tau^E - \rho\right] E_t \tag{40}$$

Government debt is $B_t = B_t^H + B_t^E$ and must evolve according to the balance sheet identity (16). Making taxes explicit, this gives

$$\dot{B}_t = \gamma K_t + r_t B_t - \tau^H H_t - \tau^E E_t. \tag{41}$$

Note that we allow B_t to be negative. Consolidating the aggregate firm balance sheet (39) with the households' balance sheet equation (38) yields the private sector's aggregate balance sheet,

$$\begin{array}{c|c} \text{Assets} & \text{Liabilities} \\ \hline K_t & H_t \\ B_t & E_t \end{array} \tag{42}$$

Equilibrium requires markets to clear at all times, given the fiscal policy in place. The following definition makes this precise.

Definition. *Given tax rates* (τ^H, τ^E)*, lump-sum transfers* (L^H, L^E)*, and a public debt trajectory* $(B_t)_{t\geq 0}$*, a **General Equilibrium with Fiscal Policy** (GEFP) is an interest rate trajectory* $(r_t)_{t\geq 0}$ *and a dynamic consumption-investment allocation such that*

1. *entrepreneurs and households behave optimally given* $(r_t)_{t\geq 0}$ *and the policy,*
2. *the government's budget evolves according to (41),*
3. *the debt market clears at each* $t \geq 0$*.*

Note that market clearing for private and public debt implies that the aggregate balance sheet constraint (42) holds at each $t \geq 0$. Note also that (42) pins down the initial amount of public debt B_0 and thus imposes consistency of fiscal policy. Indeed, by remembering that $K_0 = H + E$,

$$B_0 = H + L^H + E + L^E - K_0 = L^H + L^E \tag{43}$$

Hence, initial public debt must be just enough to finance initial transfers. It is possible to characterize GEFPs fairly generally, including existence and asymptotic behavior. However, this is of limited interest in our context, as the stationary welfare optimal equilibria implied by Proposition 2 can be characterized explicitly. Hence, existence follows directly from the implementation result in the next section.

5.2 Implementation

Proposition 2 has shown that the welfare optimal allocation can be individually optimal in a market environment with a particular given interest rate, and has identified the unique policy necessary to achieve this. The following proposition shows that this indeed uniquely implements the desired outcome as a GEFP.

Proposition 3. *Suppose fiscal policy follows the taxation rules (33)-(36) and the debt policy* $B_t = (L^H + L^E)e^{g^* t}$ *for all* $t \geq 0$*, where* g^* *is given by (15). Then,* $(r_t)_{t\geq 0}$ *is an equilibrium interest rate trajectory if and only if* $r_t = r^* = \mu - \sigma^2 x^*$ *for all* $t \geq 0$*.*

Proposition 2 and the "if"-part of Proposition 3 imply that the fiscal policy of Proposition 3 implements the constrained welfare optimum of Proposition 1 as a general equilibrium outcome. The "only-if" part of Proposition 3 states that this implementation is unique. The proof, which is given in Appendix A.2, relies on the dynamic structure of the problem and uses the Picard-Lindelöf uniqueness theorem from the theory of ordinary differential equations.

It is worth emphasizing the strong double uniqueness result here. By Proposition 2 there is exactly one fiscal policy to implement the optimum. By Proposition 3 the general equilibrium of the resulting market economy is unique. Proposition 3 is the counterpart of the Second Welfare Theorem in Arrow-Debreu economies for our incomplete markets environment. Note that lump-sum transfers at $t = 0$ alone are insufficient: optimal implementation also needs ongoing taxation or subsidization of households and entrepreneurs, together with an optimal public debt trajectory.

We now have the following result on the relationship between optimal risk-sharing, incentive provision, and public debt in the GEFP in incomplete markets.

Proposition 4. *When $\sigma > 0$, the implementation of the welfare optimum requires public debt to be strictly positive:*

$$B_0 = \frac{1 - \eta x^*}{x^*} K_0 > 0. \tag{44}$$

Proof. Evaluating (43) for (35) and (36) yields the identity in (44). The polynomial determining x^* in (14) is increasing, and strictly positive for all $x \geq 1/\eta$. Hence, $x^* < 1/\eta$, implying a strictly positive B_0. □

Hence, the optimal GEFP is not compatible with balanced budgets, and a government wishing to implement this optimum through fiscal policy must issue a positive amount of public debt. As we discuss in the next section, the reason is that the private sector does not issue sufficient debt due to missing risk-sharing opportunities. Note that the welfare optimum (x^*, χ^*) of Proposition 1 is independent of the government expenditure coefficient γ, while the taxes needed to implement it are not.

5.3 Pareto Optimality

The optimal allocation (13)-(14) is parameterized by the welfare weight η. In order to characterize the (constrained) Pareto frontier between entrepreneurs and households, remember that the common incentive-risk control variable x is implemented by a variable that resembles the entrepreneurs' common debt-equity ratio in the decentralized economy. In fact, from (25), their optimal debt-equity ratio is

$$\frac{d_t^i}{e_t^i} = \frac{k_t^i - e_t^i}{e_t^i} = x_t - 1$$

iff $k_t^i \geq e_t^i$.[26] Otherwise, it is 0 (and the entrepreneur holds debt on the asset side of the balance sheet). Note that $x_t > 1$ if and only if $D_t^H - B_t^E > 0$, i.e. if entrepreneurs are net borrowers.[27] For simplicity of exposition, we often refer to x_t as "firm leverage".

[26] In welfare optimal equilibria, entrepreneurs' optimal debt-to-equity ratios are not only identical, but even time-independent, which is the second equality in (29).

[27] By (39) and (42), entrepreneurs can only be aggregate lenders if $B_t > H_t$, which means that public debt exceeds the total wealth of households. This is rarely the case in practice, hence $x_t - 1$ can indeed be viewed as a measure of leverage.

To mirror $x_t = K_t/E_t$ we therefore introduce the variable $h_t = H_t/E_t$ to describe the Pareto conflict between entrepreneurs and households. While x_t is informative about risk and return of the productive sector, h_t is informative about the relative wealth of the household sector. By direct substitution of (10) into (18) we have

$$h_t = \chi x_t/\rho = (1-\eta)x_t \tag{45}$$

An advantage of this parametrization is that it yields a simple normalization of the economy's aggregate balance sheet (42):

Assets	Liabilities
x_t	h_t
$1 + h_t - x_t$	1

Hence, public debt is positive iff $1 + h_t - x_t > 0$. The Pareto frontier describing the welfare conflict between households and entrepreneurs can now be graphically represented in (x, h)-space by eliminating η between (13), (14), and (45) and using the fact that welfare optimal equilibria are stationary. This yields

$$h(x) = x - \frac{\rho}{\rho + \sigma^2 x^2} \tag{46}$$

for $x \geq x_{min}$, where x_{min} is the lower bound below which h would be negative and is given by the unique root of

$$\sigma^2 x^3 + \rho x - \rho = 0. \tag{47}$$

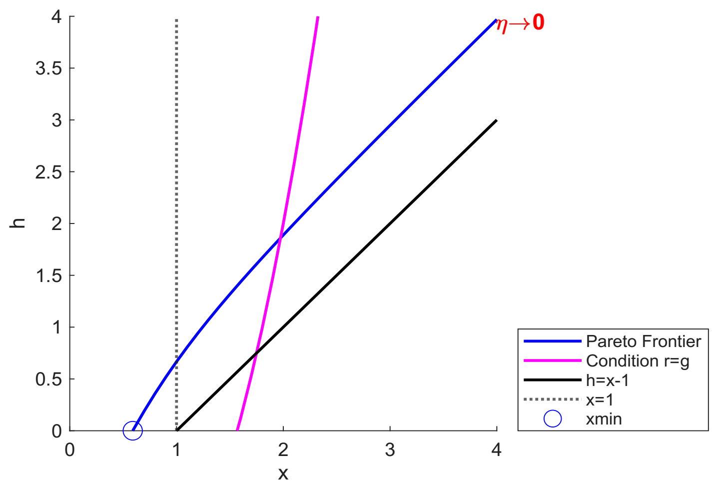


Figure 1: The Pareto Frontier in (x, h) space for $\rho = 0.01, \sigma = 0.2, \gamma = 0.05$.

Figure 1 shows the Pareto Frontier in the (x, h)-plane, for some specific values of ρ, σ, and γ. The figure also shows the "Zero-Debt-Line" $h + 1 - x = 0$, below which the allocation involves negative public debt, and the locus of points for which $r = g$, which we will discuss in detail in Section 7. The Zero-Debt-Line corresponds to the unconstrained Pareto Frontier: when there are no frictions, idiosyncratic risks can be completely eliminated, which is equivalent to taking $\sigma = 0$. In this case, optimal public debt is zero.

When $\sigma > 0$, the Pareto Frontier lies entirely above the Zero-Debt-Line, and it converges to the diagonal $h = x$ for $x \to \infty$. A simple inspection shows that the lower bound for the Pareto Frontier satisfies $x_{min} < 1$. Hence, there are Pareto Optima with $K^* < E^*$, i.e. in which entrepreneurs are net lenders on average. This means that the situation mentioned in footnote 27 can not only occur, but can even be optimal. This is the case if η is large, i.e. if fiscal policy caters strongly to the interests of entrepreneurs.

Pareto optimal equilibria are stationary, and taxation and redistribution ensure that the wealth of entrepreneurs and of households increases at the same rate on average. But levels are different, as a function of η; thus fiscal policy has a permanent redistributionary effect. In fact, the above analysis implies that there is $\bar{\eta} \in (0, 1)$ such that $H_0 > E_0 \Leftrightarrow \eta < \bar{\eta}$. More generally, the higher a group's weight in the social welfare function, the higher its steady state wealth level on average, despite the investment distortions that this entails.

However, there is a second source of heterogeneity in this economy, which leads to a potential second Pareto conflict. This heterogeneity arises from the growing inequality among entrepreneurs. Indeed, suppose (as we do) that all entrepreneurs start out with equity $e^i_0 = E_0$ at time 0, and that, in an equilibrium given by x, their aggregate equity grows at the economy's optimal growth rate g. Then, by the standard theory of Brownian motion, individual equity e^i_t at time t as given by (23) is log-normally distributed with mean and variance

$$\begin{aligned} E[e_t] &= E_0 \exp gt \\ var(e_t) &= E_0^2 \left[\exp 2gt\right] \left[\exp\left(\sigma^2 x^2 t\right) - 1\right]. \end{aligned}$$

Thus the coefficient of dispersion of entrepreneurial wealth grows over time:

$$\frac{\sqrt{var(e_t)}}{E[e_t]} = \sqrt{\exp(\sigma^2 x^2 t) - 1}.$$

The heterogeneity of entrepreneurs is endogenous in our economy: even if the initial redistribution of capital equalizes initial wealth among them, the impossibility to tax individual profits implies that the coefficient of dispersion of the distribution of their equity wealth necessarily grows and even goes to infinity. In equilibrium, in the longer run, there must be some very rich entrepreneurs with very large firms, and some who are doing much worse than the representative household, with small firms and little income.

At first sight this finding is puzzling. As discussed in Section 3.3, agents need to be exposed to instantaneous idiosyncratic risk to have incentives not to misreport output. Over time, these exposures add up to wealth inequalities and therefore to increasing inequality in realized consumption. May it be welfare-improving to have mean-reversion in wealth dynamics by redistributing from rich to poor entrepreneurs, for example with a progressive wealth tax? As in the classic work by Thomas and Worrall (1990) and Atkeson and Lucas (1992), the answer is no: incentives are more important for welfare maximization than ex-post wealth inequalities. In the present model this is less surprising than in the precursor papers, because the right incentives promote growth, which makes everybody better off. The key insight is that with leveraged finance, providing insurance stimulates risky investment and with suitable re-distribution everybody is better off. A difficult question is how to allocate those costs and benefits over time, but this is particular simple with logarithmic utility, because costs and benefits must accrue instantaneously. With more general CRRA utility the interplay between effort incentives, growth, risk aversion, and intertemporal substitution elasticities is more complex.

Hence, the increasing inequality in the welfare optimal equilibrium does not necessarily translate into immiseration, as in the dynamic insurance problems of Thomas and Worrall (1990) or Atkeson and Lucas (1992). As a reminder, immiseration means that the wealth of agents converges to 0 with probability 1 (hence, agents will become destitute in the long run almost surely, and only a very few very lucky agents become very rich). Our model is one of endogenous growth, and the incentive problem addressed by fiscal policy concerns the inefficient allocation of capital for future growth. This implies that the inequality created by dynamic insurance provision through public debt has an important further benefit: the improvement of dynamic productive efficiency and therefore value creation. If production is sufficiently productive, this overcompensates the immiseration logic. Formally, we have the following result.

Proposition 5 (Immiseration). *The equilibrium allocation of the GEFP identified in Proposition 3 does not lead to immiseration if and only if* $g > \frac{1}{2}\sigma^2 x^{*2}$. *In this case,*

$$Prob(e_t^i \leq E_0) \to 0 \quad \text{for all } i \quad \text{and } t \to \infty \tag{48}$$

However, if $g < \frac{1}{2}\sigma^2 x^{*2}$, $e_t^i \to 0$ *for* $t \to \infty$ *almost surely for every* i.

Proof. Using (29), (31), and (32) in (24) yields the following expression for the equity of entrepreneur i in equilibrium:

$$e_t^i = E_0 \exp\left((g - \frac{1}{2}\sigma^2 x^2)t + \sigma x z_t^i \right) \tag{49}$$

This is the exponential of a standard Brownian motion. If $g > \frac{1}{2}\sigma^2 x^2$, this Brownian motion goes to ∞ for $t \to \infty$ almost surely, and so does the exponential. By contradiction this directly implies (48).

If $g < \frac{1}{2}\sigma^2 x^2$, the drift of the Brownian motion is negative, the Brownian goes to $-\infty$ and $e_t^i \to 0$ almost surely. □

Remember from (15) and (14) that at the welfare optimum,

$$\begin{aligned} g - \frac{1}{2}\sigma^2 x^2 &= \mu - \gamma - \rho - \frac{1}{2}\sigma^2 x^2 - \frac{\rho\sigma^2 x}{\rho + \sigma^2 x^2} \\ &= \mu - \gamma - \rho - \frac{(1+2\eta)\rho\sigma^2 x}{2\eta(\rho + \sigma^2 x^2)} \end{aligned} \tag{50}$$

Hence, immiseration does not occur iff $\mu - \gamma$ is sufficiently large. In this case, (48) even shows that in equilibrium, in the long run every entrepreneur is almost surely better off than at $t = 0$. For households, there is nothing to show, because their wealth is not exposed to risk and grows deterministically at the constant rate g. However, Proposition 5 also shows that if $\mu - \gamma$ is small, the drift term in (49) becomes negative, and $e_t^i \to 0$ almost surely. Hence, all entrepreneurs become arbitrarily poor with probability 1, as in Thomas and Worrall (1990) and Atkeson and Lucas (1992).

While in the present model taxes and lump-sum redistribution make Pareto improvements, but do not resolve the problem of increasing inequality, the issuance of public debt helps helps unequivocally, all entrepreneurs alike, regardless of their fortunes along their stochastic individual growth paths. This is a further difference of our result from Atkeson and Lucas (1992), where a social planner does not have access to debt nor taxes.

The following thought experiment illustrates this basic benefit of public debt, by explicitly constructing the Pareto improvement that is possible in a situation where the government uses taxes optimally but issues no debt. Without loss of generality, we restrict attention to steady states. As discussed above, under balanced budgets (BB), $h = x - 1$, which by (45) implies

$$\chi = \rho\frac{x-1}{x}.$$

Hence, the welfare function (12) becomes

$$\begin{aligned} \rho W_{BB} &= \log K_0 + \frac{\mu - \gamma - \chi}{\rho} - \frac{1}{x} + \eta(\log\frac{\rho}{x} - \frac{\sigma^2 x^2}{2\rho}) + (1-\eta)\log\chi \\ &= \log K_0 + \frac{\mu - \gamma}{\rho} - 1 + \log\rho - \log x + (1-\eta)\log(x-1) - \eta\frac{\sigma^2 x^2}{2\rho} \end{aligned}$$

Maximizing W_{BB} yields the maximum welfare that can be achieved without issuing public debt. The unique maximizer $x_{BB} > 1$ is given by the first-order condition

$$\frac{\sigma^2}{\rho}x^3 - \frac{\sigma^2}{\rho}x^2 + x - \frac{1}{\eta} = 0.$$

Note the similarity with the equation defining x^*, (14), and that private leverage is higher here: $x_{BB} > x^*$. It has to partially compensate the missing public debt. x_{BB} corresponds to the following allocation of initial wealth: $E_0 = \frac{K_0}{x_{BB}}$ and $H_0 = K_0 - E_0$. The growth rate of output in this restricted optimum is $g_{BB} = \mu - \gamma - \rho$, higher than in the Pareto Optimum, and up to additive constants, the expected continuation utilities can be written as follows:

$$\begin{aligned}\rho V_{BB}^H &= \log H_0 - H_0 - E_0\\ \rho V_{BB}^E &= \log E_0 - H_0 - E_0 - \frac{\sigma^2}{2\rho E_0^2}.\end{aligned}$$

To understand how this allocation can be Pareto improved, suppose that the government issues a small amount of debt and distributes it to the two categories of agents, so that both ΔE_0 and ΔH_0 are positive. The government also adjusts the tax rates, so that the economy remains in the new steady state. The first order change in households' utility is such that

$$\rho \Delta V_{BB}^H = \frac{\Delta H_0}{H_0} - \Delta(H_0 + E_0),$$

corresponding to the difference between the relative wealth increase and the total wealth increase (equal to new government debt), which reduces growth. The equivalent term for entrepreneurial equity is

$$\rho \Delta V_{BB}^E = \frac{\Delta E_0}{E_0} - \Delta(H_0 + E_0) + \frac{\sigma^2}{\rho E_0^3}\Delta E_0,$$

where a new term appears, corresponding to the reduction in the risk premium that follows the decrease in private leverage. Hence, it is possible to distribute the additional wealth created by the government in such a way that both types of agents benefit, as long as the following two conditions hold:

$$h_0 < \frac{\Delta H_0}{\Delta E_0} < h_0 + \frac{\sigma^2 x_0^3}{\rho}.$$

This is possible in an economy with frictions, where $\sigma > 0$. As discussed above, in a frictionless economy, the Pareto-improving role of government debt disappears. Importantly, the welfare gain from issuing a positive amount of public debt benefits households and *all* entrepreneurs in *all* instances and at *all* times. This "balance sheet effect" of public debt, as we will call it, thus operates like a rising tide that lifts all the boats. Such a universal Pareto improvement has been termed "Robust Pareto Improvement" (RPI) by Aguiar et al. (2024).

Proposition 6. *Suppose that the government balances its budget and uses initial transfers and linear wealth taxes in an optimal way. In this situation, issuing public debt constitutes a Robust Pareto Improvement in the sense of Aguiar et al. (2024).*

Indeed, although our set up is quite different from Aguiar et al. (2024), in our model, just as in theirs, government debt will "complement rather than substitute for capital in an RPI" (Aguiar et al. (2024), p. 3669). Different from their model, though, our analysis exhibits a welfare-improving function of government bonds in conjunction with, rather than as an alternative to, explicit redistribution.

5.4 Laisser-Faire

As a benchmark, this subsection briefly characterizes a fully passive government, which does not engage in fiscal policy or redistribution. Such a Laisser-Faire (LF) policy has three features: (i) $B_t = 0$ for all t (balanced budget), (ii) $L^H = L^E = 0$ (no lump-sum redistribution), and (iii) $\tau_t^H = \tau_t^E = \tau_t$ (equal taxation).

Laisser-Faire therefore implies $T_t = \tau_t(H_t + E_t) = \tau_t K_t$. Together with the balanced-budget constraint $T_t = \gamma K_t$, this implies $\tau_t = \gamma$.

Furthermore, without continuing corrective taxation, the economy is not kept in steady state. The individual optimization results (21) and (22) imply that the entrepreneurs' individual capital-to-equity ratios $\frac{k_t^i}{e_t^i} = \frac{\mu - r_t}{\sigma^2}$ are still independent of i, but they now depend on t. Hence, under LF, the economy evolves on a trajectory (x_t, h_t) that is entirely contained in the Zero-Debt-Line $x = h + 1$ and starts at $x_0 > 1$. One can show that in Laisser-Faire equilibrium (x_t, h_t) converges monotonically to $(1, 0)$.

Note that Pareto Optima are not necessarily Pareto improvements over the Laisser-Faire. If we denote the expected utility of households and entrepreneurs at the optimal $x^* = x^*(\eta)$ by $V^k(\eta)$, $k = H, E$, respectively, this follows from the following two properties, which are a consequence of (13) and (14):

$$\lim_{\eta \to 0} V^E(\eta) = \lim_{\eta \to 1} V^H(\eta) = -\infty.$$

Hence, the allocation that maximizes total welfare W is a Pareto improvement over Laisser-Faire when η is intermediary. When η is large, households strictly prefer Laisser-Faire to the welfare optimum, while firms strictly prefer Laisser-Faire when η is small.

6 Taxes and Debt

As noted in the introduction, issuing public debt and distributing it to the private sector has three effects that jointly affect the consumption and investment decisions of the private sector and feed back into each other: a balance sheet effect, an interest rate effect, and a growth effect. The balance sheet effect reduces the net leverage of entrepreneurs and increases their incentives to make risky investments. The interest rate is affected because the increased supply of bonds increases the risk-free interest rate. This partially counteracts the balance sheet effect, as the entrepreneurs have to pay higher interest on their lower net debt, and it benefits households. Finally, growth is affected because issuing public debt increases the aggregate wealth of the private sector, which stimulates aggregate consumption and reduces output growth.

It is worth emphasizing that public debt does not instantaneously "crowd out" private investment in the traditional sense (see Blanchard (2008)). "Crowding out" usually refers to the substitution of private investment by public spending, which by assumption is impossible in our model, where government expenditure is exogenous. Yet, there is "dynamic crowding out" if higher public debt reduces the growth rate of the economy and capital accumulation, which lowers investment in the long-term.

Since the wealth increases generated by public debt must directly accrue to entrepreneurs in order to trigger the balance sheet effect, it is necessary to balance them by continuously redistributing wealth from entrepreneurs to households to maintain optimal growth. Hence, there is a further consequence of public debt. Since its issuance directly benefits entrepreneurs, it must be complemented by redistribution through ongoing taxation.

6.1 Taxes

To clarify the role of taxes in our economy, consider the following thought experiment. Suppose there are no financial frictions, such that all idiosyncratic risks can be diversified away and we can effectively take $\sigma = 0$. The optimal allocation then is simply implemented by redistributing initial wealth in proportions η and $1-\eta$, having no government debt, taxing entrepreneurs and households equally at $\tau_t^E = \tau_t^H = \gamma$, and thus keeping the economy at $E_t = \eta K_t$ and $H_t = (1-\eta)K_t$ at all times, as required by (14).

Suppose now that at date 0, when aggregate capital is at the level K_0, frictions appear, such that it is not possible to eliminate idiosyncratic risks anymore and $\sigma > 0$.[28] By (44), the optimal response of the government to this shock is to issue an amount $B_0 = (\frac{1}{x^*} - \eta)K_0$ of debt and to distribute it exclusively to the entrepreneurs. Indeed, by (27), $H_0 = (1-\eta)K_0$. Together with the aggregate balance sheet identity (42), this implies

$$E_0 = \eta K_0 + B_0.$$

Thus entrepreneurs are initially the only direct beneficiaries of government in-

[28] This thought experiment corresponds to the traditional experiments in macroeconomic classics, such as Kiyotaki and Moore (1997), where a stationary equilibrium is shocked unexpectedly. The specific shock analyzed here is the same as in Di Tella (2017). In fact, quoting from his paper, introducing "an aggregate uncertainty shock that increases idiosyncratic risk in the economy ... can create balance sheet recessions." Different from Di Tella (2017), we are interested in the long-run consequences of market imperfections rather than in cyclical ones.

tervention. The following result shows that in any optimal allocation, households are subsidized afterwards through ongoing taxation.

Proposition 7. *To implement the welfare optimal allocation in general equilibrium, households must be subsidized, in the sense that they contribute less in taxes than their share of public expenditures in the social welfare function:* $\tau^H H_t < (1-\eta)\gamma K_t$ *for all* t.

Proof. Evaluating the claimed inequality at the welfare optimal values shows that it is equivalent to $\tau^H < \gamma$. This in turn follows directly from (33). □

Proposition 7 states that households contribute less than their "fair share" of ongoing public expenditures.[29]

6.2 Debt

Since the optimal issue of public debt is continuously supported by redistributive taxation, it must depend on the welfare weights in the population. We now ask how. A standard measure of government indebtedness is the debt-to-GDP ratio, which in our model is given by

$$\delta_t \equiv \frac{B_t}{dY_t} dt = \frac{1 + h_t - x_t}{\mu x_t}.$$

Evaluated at the welfare optimum, the optimal debt-to-GDP ratio is

$$\delta^* = \frac{\sigma^2 x^*}{\mu(\rho + \sigma^2 x^{*2})} \tag{51}$$

which is strictly positive by Proposition 4. Two simple observations now show how δ^* depends on the welfare weighting. First, by differentiating (14), x^* is strictly decreasing in η. Second, an inspection of (51) shows that δ^* is a strictly quasiconcave function of x^*. Hence, the optimal debt-to-GDP ratio is also single-peaked in η, which is summarized in the following proposition.

Proposition 8. *The optimal debt-to-GDP ratio is a strictly quasiconcave function of the political weight* η *of entrepreneurs, with maximum at* $\widehat{\eta} = \min(1, \frac{\sigma}{2\sqrt{\rho}})$. *It converges to* 0 *for* $\eta \to 0$.

Proof. Differentiating (51) shows that δ^* as a function of x is strictly quasiconcave, with maximum at $x = \sqrt{\rho}/\sigma$. An inspection of (47) shows that $x_{\min} \geq \sqrt{\rho}/\sigma$ if and only if $\sqrt{\rho}/\sigma \leq \frac{1}{2}$. Since $x^* \in [x_{\min}, \infty)$, this shows that δ^*, as given by (51), is strictly decreasing in x^* if $\sqrt{\rho} \leq \frac{\sigma}{2}$ and strictly quasiconcave with maximum at $\sqrt{\rho}/\sigma$ otherwise. The rest of the proposition follows because $\frac{dx^*}{d\eta} < 0$ and by inserting $x^* = \sqrt{\rho}/\sigma$ into equation (14). □

[29] The proposition does not say that $\tau^{H*} < 0$. However, the proposition implies that this is the case if γ is sufficiently small. In this case, households receive continuous subsidies.

Hence, as long as the weight of entrepreneurs in the welfare function is not too large, an increase of this weight increases the debt-to-GDP ratio. This is mainly driven by the balance sheet effect discussed in the previous section: public debt, whether held directly by entrepreneurs or indirectly, when held by the household sector, reduces risk for entrepreneurs and thus stimulates investment. The more entrepreneurs matter, the more useful is debt. This effect is counteracted by the negative growth effect that takes over when entrepreneurs' interests are so dominant that further increases of debt (relative to GDP) decrease growth too much, compared to the instantaneous creation of wealth.

In our model, firm debt is safe because steady state equity follows a geometric Brownian motion and therefore never reaches zero: entrepreneurs do not default. Hence, when the government issues public debt, it does *not* create a new type of (safe) asset: government debt is exactly as good as existing private debt. However, public debt is valuable because there is not enough private debt due to the agency problem in corporate finance. Additional public debt therefore allows entrepreneurs to reduce their risk exposure.

Of course, a necessary requirement for our analysis is the credibility of the government's promise to never default. But since the government is assumed to maximize social welfare, which is achieved in the steady state with sustainable debt issuance, there is neither a reason for the government to default nor for the private sector to believe that the government will default. Not defaulting is time-consistent for our benevolent government.[30]

6.3 Illustrations

This subsection presents some simple simulations to get a sense of the magnitudes of the endogenous variables that our model predicts. Our exogenous variables are σ, ρ, μ, γ, and η. The first four of these can be taken more or less easily from existing work, the last one is a matter of interpretation, and we vary it for our simulations.

In line with standard work in the literature, we select $\mu = 0.15$ and $\rho = 0.04$ as our benchmark case. Different values are possible (we use $\rho = 0.01$ in Figure 1). Essential government services as captured by γ are a matter of interpretation. If one recognizes that a sizable part of government expenditures in most countries are redistributionary transfers to the sick and the elderly (which we do not consider

[30]Extending our model, though, in the spirit of the seminal papers of Calvo (1988) and Cole and Kehoe (2000), one can ask nevertheless whether default can be a problem. Suppose for example that for whatever reason – for instance, coordination failures in primary debt markets –, there is a chance at some point in time t that the private sector will refuse to roll over public debt. But since the government relies on taxation of wealth, even this would not cause default. By the basic balance sheet identity, $B_t = H_t + E_t - K_t$, which is strictly smaller than $H_t + E_t$. Hence, off the equilibrium, the government can confiscate sufficient private wealth via emergency taxation to stop such a debt run in the first place.

in our model), the share of essential government services in GDP may be anything between 15 to 50 percent of GDP, depending on definition, measurement, and ideology. This yields a γ between 0.02 and 0.08.

More delicate are the values for σ. Here we can draw on various sources. Calibrations for idiosyncratic shocks have been the subject of various studies, and recent work, for instance, by Bloom et al. (2018) or Arellano et al. (2019), has provided estimates for such shocks. Bloom et al. (2018) report that the yearly variance of plant-establishment-level TFP shocks in the US in non-recession time was 0.198. As some of this risk is insurable, this is an upper bound, and we assume as a benchmark for our illustrations that less than half of the risk is not insurable.[31]

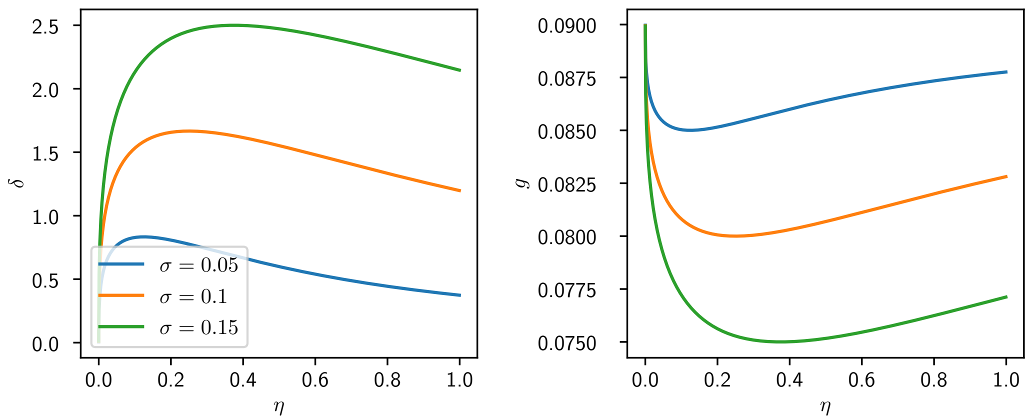
Figure 2 plots the range of predicted values for the debt-to-GDP ratio δ and the growth rate g as a function of η for different values of σ and μ. The figure also illustrates the non-monotonicity shown in Proposition 8. When $\sigma < 2\sqrt{\rho}$, we have $\widehat{\eta} < 1$ in Proposition 8, which, given the preceding discussion, seems to be an empirically relevant range in our framework.[32] In such cases, the left panels of Figure 2 display the inverse U-shape predicted in Proposition 8. Interestingly, the debt-to-GDP ratio is largest if the interests in the economy are relatively balanced, measured by values of η between 0.2 and 0.4. It decreases if one group becomes more and more dominant. The mirror image is displayed in the figures for the growth rates in the right-hand-side panels. The predictions are realistic, with a range from 25 to 200 percent for the debt-to-GDP ratios and growth rates between 2 and 9 percent in the extreme cases. Hence, the theory can justify sizeable debt levels. We also note that when the productivity of capital is lower ($\mu = 0.1$), growth rates decline, but debt-to-GDP ratios increase.

Figure 2 shows clearly that changing the weight of the productive sector (i.e., the welfare weight η) in the social welfare function changes the optimal structure of public finances and the associated tradeoffs. When this weight is small and only household interests matter, the growth rate approaches the Modified Golden Rule rate, in line with the benchmark result by Aiyagari (1994). The reason is that buffering the uninsurable productivity shocks of entrepreneurs is of little direct importance for welfare, and thus public debt is low, entrepreneurs' equity is relatively small, and private leverage is large. This implies low investment demand by entrepreneurs and thus low interest rates and high growth rates. The two effects combined yield a regime in which the growth rate exceeds the interest rate, $g > r$.

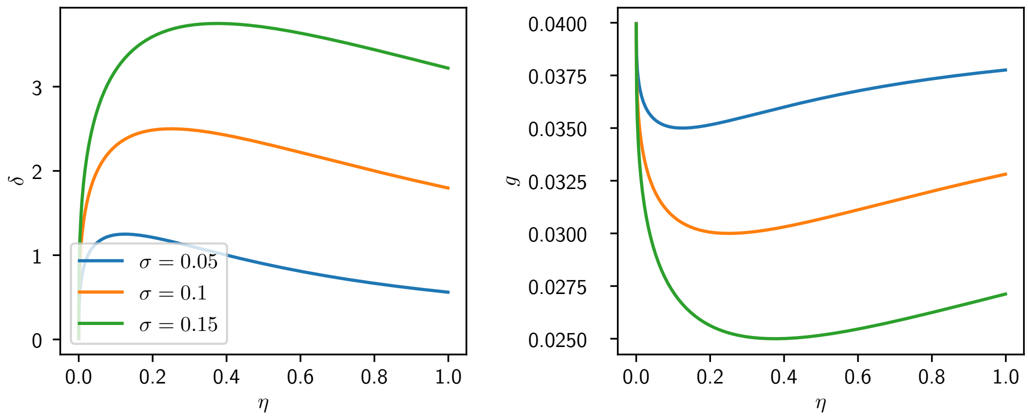
When the welfare weight of entrepreneurs is greater, buffering their shocks becomes more important. Thus, more public debt is optimally issued, equity increases, and leverage declines. Higher equity causes interest rates to rise. Greater wealth

[31] Partial insurance can be achieved through the operation of multiple plants and diversification in financial markets.

[32] For example, it comprises all combinations $\rho \geq 0.01$ and $\sigma \leq 0.2$.



$\mu = 0.15$



$\mu = 0.1$

Figure 2: Debt to GDP ratio δ and growth rate g as functions of $\eta \in (0, 1)$ for $\gamma = 0.04$ and different values of σ and μ

in the economy triggers more consumption and thus growth declines. Thus, r increases and g decreases, and the regime can switch from $r < g$ to $r > g$. However, at some level of the welfare weight of entrepreneurs the balance sheet and the risk reduction effects are dominated by the ensuing reduction of growth. If the welfare weight of entrepreneurs increases beyond that level, their losses in case of negative productivity shocks are less and less severe for them, since their leverage is low and their equity is high. Hence, it is optimal to operate with lower public debt, as this reduces current consumption by all agents and stimulates growth. In this case, the government runs an eternal primary surplus, as we will discuss in more detail in Section 7.

6.4 Funding Public Expenditure

The theory of the present paper directly speaks to a basic question of public finance: how should the government finance public spending shocks? Our model provides a surprisingly clear and simple answer.

The model describes fiscal policy with exogenous public spending needs that are considered stable over the foreseeable future. In this section we vary these spending needs. This corresponds to a thought experiment akin to the one of Section 6.1, with an unexpected shock at some date that changes long-run public spending needs and thus moves the economy from one steady state to another.[33] This is consistent with recent experience, as governments all over the globe have decided or been forced to spend large sums on security threats posed by new wars and international conflicts, or on the environmental disasters resulting from climate change and the degradation of the natural environment. Both these kinds of shocks reflect long-term challenges and seem to be more permanent.[34] Such public expenditure shocks have a straightforward interpretation in our model as an increase of γ (there could well be negative spending shocks, too, of course). For this interpretation, it is immaterial whether a higher γ provides a higher quality of public services that maintains previous living standards, or whether simply more resources are needed to maintain the public infrastructure that supports the productivity in the private sector, captured in our model by the parameter μ. [35]

Equations (13), (14), (45), (33), (34), and (51) imply that if γ changes, the

[33]Alternatively, one can think of a certain but changing path of government expenditure needs $\gamma_t K_t$. The equilibrium analysis in this paper applies directly to this variant. However, the underlying mechanism design argument for the optimality of the allocation in Biais et al. (2024) would need some modification.

[34]The same is true for the large-scale funding needs necessary to decarbonize the global economy. However, the need for drastic action has been well-known for more than 20 years, so it is more difficult to consider this as an example of an unexpected large spending shock.

[35]Alternatively, one can extend our model to directly study the link between γ and μ, if the above crisis scenarios generate such a trade-off.

policy optimum x^* remains constant, while the tax rates τ^H and τ^E needed to implement it increase one to one. This implies that debt and taxes have two very different roles in fiscal policy in our economy, which we highlight in the following proposition.

Proposition 9 (Fiscal Separation Principle)**.** *The optimal debt-to-GDP ratio is independent of the public expenditure parameter γ. Any change in government spending needs must be financed one to one by a change in tax rates.*

Hence, higher public spending needs are financed by increasing taxes, and the structural variables x and h, as well as the debt-to-GDP ratio, remain unaffected.[36] Furthermore, there is no instantaneous re-distribution after such a shock, as the aggregate private balance sheet does not need to change. This does *not* mean that a higher γ has no welfare costs. In fact, higher (exogenous) government spending crowds out private investments, which makes the economy poorer and the growth rate decline. But Proposition 9 states that public borrowing provides no structural remedy against this. The composition of the economy's aggregate balance sheet is determined by the financial market friction σ, the economic discount rate ρ, and the distributional preferences as given by η. They alone determine how the intertemporal tradeoff between wealth today and growth tomorrow stemming from market incompleteness is resolved through public borrowing. All remaining current expenses are covered by current taxes.[37]

7 Interest and Growth

Formulas (15) and (30) show how growth and interest depend on optimal fiscal policy in our model. In this section, we investigate this link and how it depends on the underlying parameters of the economy.

7.1 Interest

The following proposition follows directly from differentiating (30).

Proposition 10. *The optimal interest rate $r^* = \mu - \sigma^2 x^*$ is an increasing function of μ and η and a decreasing function of ρ. It is negative if μ or η are sufficiently small.*

[36] A similar argument has been made by Brunnermeier et al. (2021) with respect to government debt bubbles in the context of the fiscal theory of the price level.

[37] Clearly, the simplicity of our exact prescription is due to the simplicity of our model. In particular, the fact that private productivity and public expenditure enter the social welfare function only in terms of their difference $\mu - \gamma$ is due to the assumption of constant returns to scale and the lack of any direct impact of γ on μ. Similarly, the assumption of log preferences excludes non-trivial intertemporal substitution patterns coming from the demand side and more traditional tax smoothing considerations. But the separation of intertemporal market completion through public debt from the funding of current public expenditures probably is a more general insight worth remembering for broader policy prescriptions.

Proposition 10 sheds some interesting light on the recent debate about the observation that real interest rates have fallen over the last decades and have reached negative territory in a variety of industrialized countries, already before the recent inflationary hump. At the center of most explanations for this phenomenon is the observation that the amount of savings, relative to investment demand, has changed. While some explanations put emphasis on the origin of changes in savings, others put more emphasis on changes in productivity or put emphasis on both. One prominent voice is Rachel and Summers (2019), who stress that these secular movements are for a larger part a reflection of changes in saving and investment propensities. They argue that the industrialized world will probably face a longer period of secular stagnation, with sluggish growth and low real interest rates.[38]

Our results point to other structural factors that might contribute to low real interest rates. For instance, and consistent with Proposition 10, permanent shifts in the objectives of policy-making with respect to risk-bearing versus non-risk-bearing agents can induce a secular decline and even negative values of real interest rates. Proposition 10 is also consistent with the suggested link between aggregate productivity and interest rates.

7.2 The Sustainability of Fiscal Policy

The preceding results make it possible to characterize the relation between the growth and the interest rate in general equilibrium with optimal fiscal policy. Our simple model makes an explicit but non-trivial prediction about this widely debated relation.

Proposition 11. *In the Pareto optimal equilibrium, $g^* > r^*$ if and only if*

$$2\eta\left(\rho+\gamma\right)+\left(\rho+\gamma+\eta\right)\sqrt{\eta\left(1+\frac{\gamma}{\rho}\right)}<\sigma^2. \tag{52}$$

The proof of Proposition 11 is in Appendix A.3. The proposition makes precise predictions about the determinants of the difference between r and g in the welfare optimal equilibrium, involving four of the five exogenous variables of the model. As discussed in the introduction, at least in recent history, the case $g > r$ seems to have been more relevant than the opposite case. In this respect, the prediction of Proposition 11 is that the growth rate will optimally exceed the interest rate when the private propensity to consume ρ, public expenditures γ, and the political weight of entrepreneur interests η are small, and when idiosyncratic production risk σ is large. These predictions are independent of the productivity of capital, μ.

[38] For discussions (and evidence) on how to differentiate whether rising income inequality or an aging of the population can have contributed to an increase in savings see e.g. Mian et al. (2021); von Weizsäcker

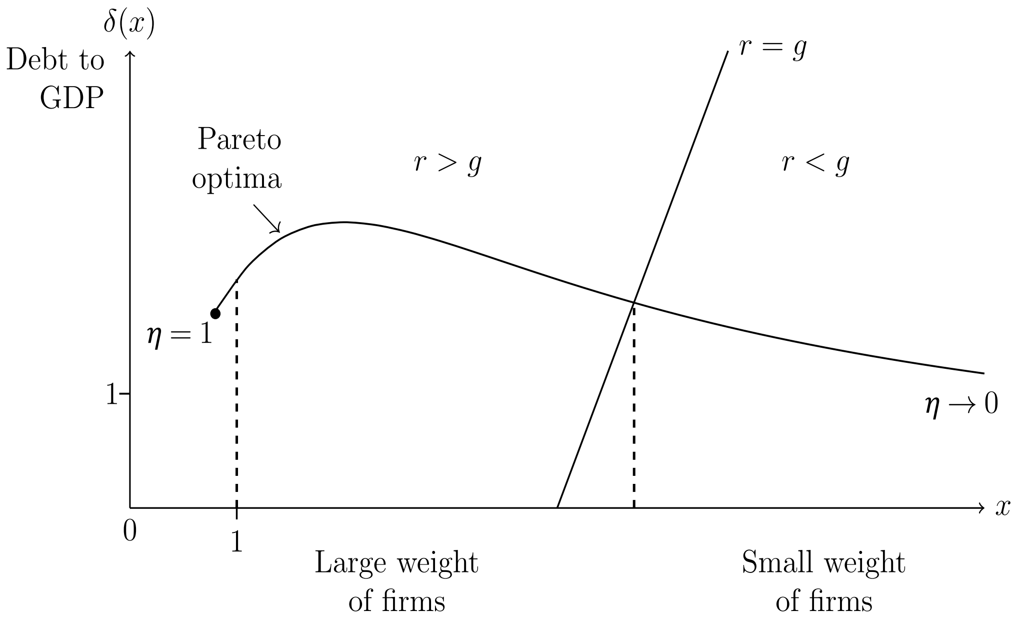


Figure 3: Regimes for parameter values $\rho = 0.04, \sigma = 0.1, \mu = 0.1$

Figure 3, which mirrors the left panels of Figures 2, illustrates the insight of Proposition 11 that depending on the welfare weight of entrepreneurs, the economy can be in different regimes $r - g > 0$ or $r - g < 0$.

Whether $r < g$ or not has been a central question in recent debates about the sustainability of the US' and other countries' fiscal policy. From an asset pricing perspective, Cochrane (2019) describes the limits of public deficits by noting that in models with infinitely-lived agents, "[t]he market value of government debt equals the present discounted value of primary surpluses." Consistent with the prediction of Proposition 11, Cochrane (2022) argues that under complete financial markets ($\sigma = 0$ in our model), a permanent relationship $r < g$ is theoretically implausible, and empirically unlikely when r and g are measured correctly.[39] On the other hand, Blanchard (2019) adopts a more positive view on the theoretical possibility of $r < g$ and investigates the potential and limitations of a fiscal expansion at little or no fiscal cost.

In our model of an economy with endogenous growth and imperfect macroeconomic risk-sharing, the return on safe debt r can fall below g. If buffering the losses of entrepreneurs has less weight in the welfare function, public debt issuance and the reduction of private leverage are less important. As a consequence, entrepreneurs are only willing to invest in risky production if the real interest rate is sufficiently

and Krämer (2019) discuss how technological progress and demography may have jointly contributed to a secular decline in real interest rates.

[39]Cochrane (2022) provides a comprehensive account how the $r < g$ – debate is connected to the fiscal theory of the price level.

low. Hence, as summarized in Figure 3 above, there is a role for government policy to actively reduce r in such cases.

Our analysis can reconcile both views about the dynamics of the government budget in a single model. The government's flow budget constraint at date t, (16), can be written as

$$\dot{B}_t = \gamma K_t + r_t B_t - T_t = rB_t - S_t, \tag{53}$$

where S_t is the primary surplus. Consider an arbitrary steady state (not necessarily optimal) and let r and g be the associated interest and growth rates, respectively. Discounting and integrating (53) between dates 0 and some later date T yields:[40]

$$B_0 = \int_0^T S_t e^{-rt} dt + B_T e^{-rT}.$$

This relation can be viewed as the balance sheet identity for the public sector, with liabilities B_0 and two types of assets as follows:

Assets	Liabilities
$X_0 := \int_0^T S_t e^{-rt} dt$	B_0
$W_0 := B_T e^{-rT}$	

where we let $T \to \infty$.

As in our previous discussion, we can distinguish two cases. The first case is $r > g$. Then W_0 tends to zero when T tends to ∞, and we obtain the standard relationship that the value of debt equals the net present value of future primary surpluses, as argued, e.g., by Cochrane (2019). The second case is $r < g$. Then W_0 tends to $+\infty$ and X_0 tends to $-\infty$. Hence, in the limit, the balance sheet identity $X_0 + W_0 = B_0$ is not well defined. However, we can interpret $B_T e^{-rT}$ as a form of intangible asset for the government, which can be attributed to its capacity to borrow again in the future and may be called government "goodwill". In fact, our analysis shows that it is rather the government's "eternal power to issue safe debt" that creates this intangible asset. As long as the government can convince investors of its capacity to sustain a high enough level of growth, this intangible asset has a positive value.

In light of the results of the last two sections and under the assumption that σ is not too large,[41] we can therefore distinguish two polar cases for the impact of entrepreneurial interests η on the sustainability of government deficits. First, if η is small, $g > r$ in equilibrium, and the government runs increasing budget deficits that

[40]Which discount rate should be used for the government budget constraint has been the subject of recent work. Brunnermeier et al. (2021) and Reis (2021) offer particular rationales for using discount rates different from r.

[41]We need the inequality in (52) to be reversed for $\eta = 1$, which implies an upper bound on σ. This is consistent with our discussion of plausible parameter ranges in Section 6.2, and in particular with the assumption $\sigma < 2\sqrt{\rho}$ that ensures $\widehat{\eta} < 1$ in Proposition 8. If σ is large (which seems implausible empirically), we have $r < g$ for all η.

it covers by taxes and by rolling over ever-increasing public debt. Nevertheless, by Proposition 8 the public debt-to-GDP ratio is small. Second, if η is large, we have $g < r$ in equilibrium, "[t]he market value of government debt equals the present discounted value of primary surpluses" (Cochrane (2019)), and the public debt-to-GDP ratio is intermediary. For medium values of η, the public debt-to-GDP ratio is large, the growth rate is low, and the sign of $r - g$ depends on γ, σ, and γ as given by (52). Hence, government deficits have a "Cochranian" interpretation or a "Blanchardian" one, depending on η.

8 Conclusion and Applications

We have presented a simple endogenous growth model in which government debt issuance affects corporate leverage, and thus the investment and growth dynamics of the economy, through changes in the mix of private and public debt. It highlights how the weights of different private agents in the government welfare function impact the relationship between r and g. In this sense, interest, growth, and public debt are a matter of redistributionary political trade-offs.

Our model calls for many extensions and allows a variety of applications. First, we have focused on a basic informational friction of outside finance, which rules out simple state-contingent finance such as outside equity. This clearly is restrictive (but standard in the macro finance literature), and should be generalized to a setting that accommodates larger firms with outside equity held by dispersed investors. There are several promising possibilities for doing this in a theoretically rigorous way. One is the approach by Biais et al. (2007) who introduce the possibility that firms hold cash reserves in a publicly verifiable bank account. If such bank accounts are costly to entertain (in the sense of period fixed costs), then the population of firms will fall into two classes - small and medium size firms that are approximated by the agency structure of the present paper, and large firms that issue public equity subject to the above costly verification procedure. Another approach uses the framework with inefficient diversion and contract termination developed by DeMarzo and Sannikov (2006), which uses credit lines as a complement to straight debt as a corporate governance mechanism. This generalization is likely to create a private sector with a richer balance sheet and allow some diversification, but open up new tradeoffs for public debt versus private assets and the resulting interest and growth dynamics.

Second, it would be desirable, in particular for empirical calibrations that go beyond Section 6.3, to add a labor market and include more standard household labor and consumption behavior. This can be done by generalizing our one-factor production function to the Cobb-Douglas form and then follow the standard production-based asset pricing literature (see, e.g., Kogan and Papanikolaou (2012)) by assuming that firms always use the optimal quantity of labor at the going wage. If one assumes that households supply labor inelastically, this yields the one-factor model of this paper as a reduced form quite easily. But in future work it would be desirable to include a real labor-leisure tradeoff and obtain more realistic labor supply functions.

Third, the present model has completely abstracted from financial intermediaries, who in fact may be able to overcome some of the agency frictions at the possible cost of lengthening the financial contracting chain (see, e.g., He and Krishnamurty (2012), Di Tella (2020) or the early version of Krueger and Uhlig (2022)). We have not done this, because in our contracting model short-term debt suffices to implement the second-best. However, we are using our model as a building block of a theory of money and banking in Gersbach et al. (2025), in which central bank reserves play a safety role for commercial banks, while controlling monetary policy, similar to the role of government debt in the current model. Since the Great Financial Crisis of 2007-2009, the reserves of commercial banks in the US, the UK, Japan, and in the Euro Area have strongly increased, albeit to different degrees. Our preliminary results support the argument that banks' holding large amounts of central bank reserves can be desirable from a welfare perspective when banks face significant uninsurable idiosyncratic risks.

Fourth, as briefly mentioned in Section 6.4, it is promising to endogenize public expenditures in our model and to address the public goods problem of government expenditure explicitly. This would make the possible tradeoff between higher γ and higher private productivity μ more transparent and would also allow to study the impact of preference shocks for public good provision on the mix of taxes and debt. In preliminary work we have found that such a theory will provide more nuanced results on the desirability of public debt.

More generally, the present paper has implications for normative macroeconomic theories in which government debt serves a socially desirable purpose. To what extent is the rise of government debt over the past decades an optimal response to changing fundamentals? For instance Yared (2019) provides a comprehensive account of political economy theories on government debt and discusses how these theories may explain a substantial part of the long-term trend in government debt accumulation. Our model would be a natural starting point to introduce political constraints such as participation constraints of firms, upper limits on taxation of firms, and lower bounds of consumption utility of households. When such constraints become binding, the balance between taxes and debt issuance may shift. Along similar lines, changes of the weight η in the welfare function can have a direct impact on the problem of inequality and immiseration addressed in Proposition 5.

A Appendix

A.1 Derivation of expected utility in Section 3.4

Under the incentive-compatible mechanism described in Section 3.3, entrepreneurs' expected utility is

$$\rho V^E = \log \frac{\rho}{x} + \rho \mathbb{E} \int_0^\infty e^{-\rho t} \log k_t^i dt$$

By the Itô-Doeblin Lemma and using (8),[42]

$$\begin{aligned} \log k_t^i &= \log k_0^i + \int_0^t \frac{1}{k_s^i} dk_s^i - \frac{1}{2} \int_0^t \frac{1}{(k_s^i)^2} \sigma^2 x^2 (k_s^i)^2 ds \\ &= \log k_0^i + \int_0^t \left(g - \frac{1}{2} \sigma^2 x^2 \right) ds + \int_0^t \sigma x dz_s^i \end{aligned}$$

Integrating partially yields

$$\rho \mathbb{E} \int_0^\infty e^{-\rho t} \log k_t^i dt = \log k_0 + \frac{1}{\rho} \left(g - \frac{1}{2} \sigma^2 x^2 \right)$$

A.2 Proof of Proposition 3

A.2.1 "If"

We must verify the three properties of a GEFP.

The first property follows directly from Proposition 2. The third property follows by the construction of (B_t): by (43) and the definition of (B_t), the aggregate balance sheet holds at time $t = 0$, and it holds for all $t > 0$ because B_t grows at the rate g^*, just as H_t, E_t, and K_t. To verify the second property, we must show that $B_t = H_t + E_t - K_t$ satisfies (41), i.e. that

$$\dot{H}_t + \dot{E}_t - \dot{K}_t = \gamma K_t + r(H_t + E_t - K_t) - \tau^H H_t - \tau^E E_t. \tag{A.1}$$

Using (28) and (31) and rearranging, (A.1) is equivalent to

$$(r - \gamma - g^*) K_t = \rho H_t + \left(\rho - \left(\frac{\mu - r}{\sigma} \right)^2 \right) E_t \tag{A.2}$$

We know from (25) that $K_t = x E_t$. Using this together with $H_t = (1 - \eta) K_t$, we can substitute out for H_t and E_t by K_t. Substituting r from (30) then shows that (A.2) is equivalent to

$$\rho - \sigma^2 x^2 - \rho \eta x + \frac{\sigma^4 x^4}{\rho + \sigma^2 x^2} = 0,$$

which holds for $x = x^*$ by (14).

[42] See, e.g., Shreve (2004), p. 187.

A.2.2 "Only If"

Suppose fiscal policy follows the taxation and transfer rules (33)-(36) and the debt policy $B_t = (L^H + L^E)e^{g^* t}$ for all $t \geq 0$. Suppose also that $(r_t)_{t\geq 0}$ is an equilibrium interest rate trajectory. We must show that $r_t = r^* = \mu - \sigma^2 x^*$ for all $t \geq 0$.

The aggregate balance sheet constraint (42) together with (37), (40), (22), and (41) implies

$$\begin{aligned}
\dot{K}_t &= \dot{H}_t + \dot{E}_t - \dot{B}_t \\
&= \left(r_t - \tau^H - \rho\right) H_t + \left(r_t - \tau^E - \rho\right) E_t + (\mu - r_t)K_t - \gamma K_t - r_t B_t + T_t \\
&= (\mu - \gamma)K_t - \rho(H_t + E_t),
\end{aligned} \tag{A.3}$$

which is the economy's IS equation (equality of investment and net savings).[43]

At each date t, the four aggregate variables K_t, B_t, E_t, H_t are linked by the balance sheet identity (42). In fact, by the homogeneity of the entrepreneurs' investment problem, ratios of the state variables are sufficient to characterize equilibrium. We pick here the capital-equity ratio x_t as defined in (25), and $h_t \equiv \frac{H_t}{E_t}$, the ratio of household wealth over entrepreneurial equity.[44] The trajectories of the two state variables (x_t, h_t) completely determine all aggregate variables (output, consumption, and investment) in equilibrium. In fact, by (A.3), the equilibrium growth rate g_t of capital is

$$g_t = \frac{\dot{K}_t}{K_t} = \mu - \gamma - \rho \frac{h_t + 1}{x_t}. \tag{A.4}$$

By (37), aggregate household wealth grows according to

$$\frac{\dot{H}_t}{H_t} = \mu - \rho - \tau^H - \sigma^2 x_t \tag{A.5}$$

and aggregate equity of entrepreneurs, similarly, according to

$$\frac{\dot{E}_t}{E_t} = \mu - \rho - \tau^E - \sigma^2 x_t(1 - x_t) \tag{A.6}$$

Given this direct relation between equilibria and the $x_t - h_t$ trajectories, we now characterize these trajectories.

The initial values of the system are given by the lump sum transfers at date 0:

$$h_0 = \frac{H_0}{E_0} = \frac{\tilde{H} + L^H}{\tilde{E} + L^E}, \tag{A.7}$$

$$x_0 = \frac{K_0}{E_0} = \frac{\tilde{H} + \tilde{E}}{\tilde{E} + L^E}. \tag{A.8}$$

The dynamics of the state variables for $t > 0$ are then determined by the instantaneous tax rates. Using the definition of h_t, (A.5) and (A.6) imply

$$\dot{h}_t = (\tau^E - \tau^H - \sigma^2 x_t^2) h_t. \tag{A.9}$$

[43] (A.3) is the counterpart of the optimality condition (7) in the mechanism design problem.

[44] See the motivation and discussion in Section 5.3.

Similarly, using (A.4) and (A.6),

$$\dot{x}_t = (\sigma^2 x_t^2 - \rho)(1 - x_t) + (\tau^E - \gamma)x_t - \rho h_t. \tag{A.10}$$

If the system (A.7)-(A.10) has a solution that stays in the interior of the positive (x, h) quadrant, then this solution yields a unique general equilibrium for the given fiscal policy, as shown above. Using (33)-(36) and the definition of x^* in (14), one easily verifies that the constant trajectory $(x_t, h_t) = (x^*, (1-\eta)x^*)$ solves (A.7)-(A.10). By the Picard-Lindelöf Theorem from the theory of ordinary differential equations (see, e.g., Hirsch and Smale (1974)), the system only has one solution, which is maximal. Hence, the interest rate trajectory $(r_t)_{t\geq 0}$ we started out with must be the constant trajectory $r_t = r^*$.

A.3 Proof of Proposition 11

From (15) and (30) we have

$$r^* - g^* = \frac{\rho}{x^*} - \sigma^2 x^* + \rho(1-\eta) + \gamma.$$

Using (14), this implies

$$\frac{x^{*2}}{\rho}(r^* - g^*) = \left(1 - \eta + \frac{\gamma}{\rho}\right) x^{*2} + 2x^* - \frac{1}{\eta}.$$

Hence, we have $r^* < g^*$ iff $x^* < \widetilde{x}$, where $\widetilde{x}$ is the unique positive solution to

$$x^2 + \frac{2}{y}x - \frac{1}{\eta y} = 0, \tag{A.11}$$

i.e.

$$\widetilde{x} = \frac{1}{y}\left[\sqrt{1 + \frac{y}{\eta}} - 1\right],$$

where $y \equiv 1 - \eta + \frac{\gamma}{\rho}$. Using the definition of x^* and (A.11), the condition $x^* < \widetilde{x}$ is equivalent to

$$\left(4\eta + y + \frac{\rho\eta}{\sigma^2}y^2\right)\left[\sqrt{1 + \frac{y}{\eta}} - 1\right] > 2y + \frac{\rho}{\sigma^2}y^3. \tag{A.12}$$

In a number of straightforward steps, (A.12) can be re-written as (52) in the proposition.

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Internet Appendix: The Individual Decision Problems of Section 4.2

For completeness, this appendix provides a detailed solution to the individual optimization problems of Section 4.2 that were only sketched in the main text.

Households

Suppose that the representative household has initial net worth n_0^H at time $t = 0$, no further income later, and can only save via safe debt. Consider the variation of the household's decision problem in which the household starts out at time $t \geq 0$ with net worth $n > 0$. It chooses a consumption path c_s^H, $s \geq t$, to solve the standard consumption problem

$$\begin{aligned}
&\max_{c^H} \int_t^\infty e^{-\rho s} \log c_s^H ds \\
&dn_s^H = \left((r_s - \tau_s^H) n_s^H - c_s^H\right) ds \\
&n_t^H = n \\
&n_s^H \geq 0.
\end{aligned} \tag{B1}$$

Denote the optimal consumption path for this problem by $c_s^H(t, n)$.

Remark 1. *The problem is homogeneous and invariant to scaling. Hence, if $c_s^H = c_s^H(t, n)$, $s \geq t$, is an optimal path for the problem with initial condition $n_t^H = n$, then αc_s^H, $s \geq t$, is an optimal path for the problem with initial condition $n_t^H = \alpha n$, for $\alpha > 0$.*

Hence, any optimal path satisfies

$$c_s^H(t, n) = c_s^H(t, 1) n.$$

Let $V^H(t, n)$ be the value function of the problem. Homogeneity implies

$$\begin{aligned}
V^H(t, n) &= \int_t^\infty e^{-\rho s} \log c_s^H(t, n) ds \\
&= \frac{e^{-\rho t}}{\rho} \log n + v^H(t),
\end{aligned} \tag{B2}$$

where

$$v^H(t) = \int_t^\infty e^{-\rho s} \log c_s^H(t, 1) ds \tag{B3}$$

is independent of n.

Ignoring the non-negativity conditions (which will be satisfied at the optimum), the Bellman Equation of the household's problem is

$$\frac{\partial V^H}{\partial t} + \max_c \left[e^{-\rho t} \log c + \frac{\partial V^H}{\partial n} \left((r_t - \tau_t^H) n - c \right) \right] = 0.$$

From (B2), we have

$$\frac{\partial V^H}{\partial n} = \frac{e^{-\rho t}}{\rho n},$$

such that the Bellman Equation becomes

$$\frac{\partial V^H}{\partial t} + \max_c \left[e^{-\rho t} \log c + \frac{e^{-\rho t}}{\rho n} \left((r_t - \tau_t^H) n - c \right) \right] = 0. \tag{B4}$$

It is easy to see that the first-order condition

$$c = \rho n \tag{B5}$$

is necessary and sufficient for the maximization problem in (B4). In particular, (B5) implies that $c > 0$. The Bellman Equation thus is equivalent to

$$-e^{-\rho t} \log n + \dot{v}^H(t) + e^{-\rho t} \left[\log \rho n - 1 + \frac{r_t - \tau_t^H}{\rho} \right] = 0,$$

which is equivalent to

$$\dot{v}^H(t) = \frac{e^{-\rho t}}{\rho} \left[\rho - \rho \log \rho - r_t + \tau_t^H \right].$$

This can be integrated explicitly to yield

$$\rho v^H(t) = (1 - \log \rho) \left(1 - e^{-\rho t} \right) - \int_0^t e^{-\rho s} (r_s - \tau_s^H) ds + \rho v^H(0). \tag{B6}$$

By (B5), if $n_s^H(t, n)$ is on the trajectory generated by $c_s^H(t, n)$, $s \geq t$, the optimal policy is

$$c_s^H(t, n) = \rho n_s^H(t, n). \tag{B7}$$

Hence, inserting (B7) into (B1) yields the law of motion for household savings with initial value 1 at time $t = 0$, $n_s^H(0, 1)$, as

$$\frac{d n_s^H(0, 1)}{ds} = (r_s - \tau_s^H - \rho) n_s^H(0, 1).$$

Integrating yields

$$\log n_s^H(0, 1) = \int_0^s (r_\tau - \tau_\tau^H - \rho) d\tau, \tag{B8}$$

where the constant of integration in (B8) is $\log n_0^H(0, 1) = \log 1 = 0$, by the construction of v.

Inserting (B7) and (B8) into (B3) yields, for $t = 0$,

$$\begin{aligned} v^H(0) &= \int_0^\infty e^{-\rho s}(\log \rho + \log n_s^H(0,1))ds \\ &= \frac{\log \rho}{\rho} + \int_0^\infty e^{-\rho s} \int_0^s (r_\tau - \tau_\tau^H - \rho) d\tau ds \\ &= \frac{\log \rho}{\rho} - \frac{1}{\rho} + \frac{1}{\rho} \int_0^\infty e^{-\rho \tau}(r_\tau - \tau_\tau^H) d\tau. \end{aligned}$$

Combining this with (B6) yields

$$\rho v^H(t) = -(1 - \log \rho) e^{-\rho t} + \int_t^\infty e^{-\rho s}(r_s - \tau_s^H) ds,$$

which together with (B2) yields the households' value function as

$$\rho V^H(t,n) = e^{-\rho t} (\log(\rho n) - 1) + \int_t^\infty e^{-\rho s} \left(r_s - \tau_s^H\right) ds.$$

Entrepreneurs

Net of initial lump sum taxes, at time $t = 0$ entrepreneur i has an initial equity position $e_0^i > 0$. Consider the variation where she starts at time t with equity $e^i > 0$. She chooses a path k_s^i, e_s^i, c_s^i, $s \geq t$ such as to

$$\begin{aligned} &\max_{k^i, e^i, c^i} \mathbb{E} \int_t^\infty e^{-\rho s} \log c_s^i ds \\ &de_s^i = \left[(\mu - r_s)k_s^i + (r_s - \tau_s^E)e_s^i - c_s^i\right] ds + \sigma k_s^i dz_s^i \qquad \text{(B9)} \\ &e_t^i = e^i \\ &e_s^i \geq 0, \end{aligned}$$

where equation (B9) is the flow of funds equation (20) in the main text. Denote the value function of the problem by $V^E(t, e^i)$.

Since, as in the household problem, the feasible set is homogeneous, any solution is invariant to scaling, and we must have, at the optimum,

$$(k_s^i(t, e^i), c_s^i(t, e^i)) = (k_s^i(t,1)e^i, c_s^i(t,1)e^i).$$

Therefore,

$$V^E(t, e^i) = \frac{e^{-\rho t}}{\rho} \log e^i + v^E(t), \qquad \text{(B10)}$$

where

$$v^E(t) = \mathbb{E} \int_t^\infty e^{-\rho s} \log c_s^i(t,1) ds \qquad \text{(B11)}$$

is independent of e^i.

We first solve the unconstrained problem, in which we ignore the non-negativity constraint on e_s^i. In this case, the Bellman Equation is

$$\frac{\partial V^E}{\partial t} + \max_{k,c}\left[e^{-\rho t}\log c + \frac{\partial V^E}{\partial e}\left((\mu - r_t)k + (r_t - \tau_t^E)e^i - c\right) + \frac{\partial^2 V^E}{\partial e^2}\frac{\sigma^2}{2}k^2\right] = 0.$$

From (B10), we have

$$\begin{aligned} \frac{\partial V^E}{\partial e} &= \frac{e^{-\rho t}}{\rho e^i} \\ \frac{\partial^2 V^E}{\partial e^2} &= -\frac{e^{-\rho t}}{\rho (e^i)^2}. \end{aligned}$$

The Bellman Equation therefore becomes

$$\frac{\partial V^E}{\partial t} + max_{k,c}e^{-\rho t}\left[\log c + \frac{1}{\rho e^i}\left((\mu - r_t)k + (r_t - \tau_t^E)e^i - c\right) - \frac{1}{2\rho (e^i)^2}\sigma^2 k^2\right] = 0 \tag{B12}$$

and the first-order conditions

$$c = \rho e^i \tag{B13}$$

$$k = \frac{\mu - r_t}{\sigma^2}e^i \tag{B14}$$

are necessary and sufficient for the maximum in (B12). In particular, (B13) implies that $c > 0$. The Bellman Equation therefore is equivalent to

$$\begin{aligned} -e^{-\rho t}\log e^i + \dot{v}^E(t) + e^{-\rho t}\left[\log \rho e^i - 1 + \frac{r_t - \tau_t^E}{\rho} + \frac{(\mu - r_t)^2}{2\rho\sigma^2}\right] &= 0 \\ \Leftrightarrow \dot{v}^E(t) = e^{-\rho t}\left[1 - \log\rho - \frac{r_t - \tau_t^E}{\rho} - \frac{(\mu - r_t)^2}{2\rho\sigma^2}\right]. \end{aligned}$$

This is a deterministic ODE that can be integrated explicitly to yield

$$\rho v^E(t) = (1 - \log\rho)\left(1 - e^{-\rho t}\right) - \int_0^t e^{-\rho s}\left(r_s - \tau_s^E + \frac{(\mu - r_s)^2}{2\sigma^2}\right)ds + \rho v^E(0) \tag{B15}$$

From (B13)–(B14), if $e_s^i = e_s^i(t, e^i)$ is on a trajectory generated by $c_s^i(t, e^i)$ and $k_s^i(t, e^i)$, $s \geq t$, the optimal policy is

$$c_s^i(t, e^i) = \rho e_s^i \tag{B16}$$

$$k_s^i(t, e^i) = \frac{\mu - r_s}{\sigma^2}e_s^i. \tag{B17}$$

Hence, inserting (B16) and (B17) into the equation of motion (B9) yields the (random) law of motion for entrepreneur equity, with $s \geq t$ and $e_t^i = e_t^i(t, e^i) = e^i$, as

$$de_s^i = \left[\left(\frac{\mu - r_s}{\sigma}\right)^2 + r_s - \tau_s^E - \rho\right]e_s^i ds + \frac{\mu - r_s}{\sigma}e_s^i dz_s^i \tag{B18}$$

$$\equiv (\beta_s - \rho)e_s^i ds + \gamma_s e_s^i dz_s^i, \tag{B19}$$

where we have set, for simplicity,

$$\beta_s = \left(\frac{\mu - r_s}{\sigma}\right)^2 + r_s - \tau_s^E \tag{B20}$$

$$\gamma_s = \frac{\mu - r_s}{\sigma}. \tag{B21}$$

We must determine $v^E(0)$. From (B11), using (B17), we have

$$\begin{aligned} v^E(0) &= \mathbb{E}\int_0^\infty e^{-\rho s} \log \rho e_s^i(0,1) ds \\ &= \frac{\log \rho}{\rho} + \mathbb{E}\int_0^\infty e^{-\rho s} \log e_s^i(0,1) ds. \end{aligned} \tag{B22}$$

Applying the Itô-Doeblin formula (Shreve (2004), p. 187) to (B19) yields

$$\begin{aligned} d \log e_s^i &= \frac{1}{e_s^i} de_s^i - \frac{1}{2\left(e_s^i\right)^2} \gamma_s^2 \left(e_s^i\right)^2 ds \\ &= \left(\beta_s - \rho - \frac{1}{2}\gamma_s^2\right) ds + \gamma_s dz_s^i \end{aligned}$$

For $e_s^i = e_s^i(0,1)$, where by definition $e_0^i = 1$, this means with probability 1,

$$\log e_s^i(0,1) = \int_0^s \left(\beta_\tau - \rho - \frac{1}{2}\gamma_\tau^2\right) d\tau + \int_0^s \gamma_\tau dz_\tau^i.$$

By the definition of the stochastic integral, under standard integrability assumptions for r_s, the expectation in (B22) then is

$$\begin{aligned} \mathbb{E}\int_0^\infty e^{-\rho s} \log e_s^i(0,1) ds &= \int_0^\infty e^{-\rho s} \int_0^s \left(\beta_\tau - \rho - \frac{1}{2}\gamma_\tau^2\right) d\tau ds \\ &= -\frac{1}{\rho} + \int_0^\infty e^{-\rho s} \int_0^s \left(\beta_\tau - \frac{1}{2}\gamma_\tau^2\right) d\tau ds \\ &= -\frac{1}{\rho} + \frac{1}{\rho}\int_0^\infty e^{-\rho \tau} \left(\beta_\tau - \frac{1}{2}\gamma_\tau^2\right) d\tau. \end{aligned}$$

Inserting this into (B22) and using (B20)–(B21),

$$\rho v^E(0) = \log \rho - 1 + \int_0^\infty e^{-\rho s} \left(r_s - \tau_s^E + \frac{(\mu - r_s)^2}{2\sigma^2}\right) ds.$$

Combining this with (B15) yields

$$\rho v^E(t) = -e^{-\rho t}(1 - \log \rho) + \int_t^\infty e^{-\rho s} \left(r_s - \tau_s^E + \frac{(\mu - r_s)^2}{2\sigma^2}\right) ds. \tag{B23}$$

Finally, inserting (B23) into the value function (B10), yields

$$\rho V^E(t, e^i) = e^{-\rho t}(\log \rho e^i - 1) + \int_t^\infty e^{-\rho s} \left(r_s - \tau_s^E + \frac{(\mu - r_s)^2}{2\sigma^2}\right) ds.$$