Optimal Allocations in Growth Models with private Information*

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Abstract

This paper considers a class of growth models with idiosyncratic human capital risk and private information about individual effort choices (moral hazard) and private information about individual shock histories (adverse selection). Households are infinitely-lived and have preferences that allow for a time-additive expected utility representation with a one-period utility function that is additive over consumption and effort as well as logarithmic over consumption. Human capital investment is risky due to idiosyncratic shocks that follow a Markov process with transition probabilities that depend on effort choices. The production process is represented by an aggregate production function that uses physical capital and human capital as input factors. We show that constrained optimal allocations are simple in the sense that individual effort levels and individual consumption growth rates are history-independent. Further, constrained optimal allocations are the solutions to a recursive social planner problem that is simple in the sense that exogenous shocks are the only state variables.

Keywords: Economic Growth, Private Information, Human Capital Risk

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1. Introduction

Dynamic models with private information about individual effort (moral hazard) or shock histories (adverse selection) have been studied by a large literature in macroeconomics and public finance (Ljungqvist and Sargent, 2018, and Stantcheva, 2020). In these models, constrained optimal allocations often display a dependence on individual histories rendering the analysis of even simple economic problems a challenging task. The literature has tried to circumvent this tractability problem using a recursive approach with additional endogenous state variables (promised utility), but this approach quickly reaches its computational limits when studying economies with multidimensional investment choices or aggregate shocks. Moreover, most applied work has confined attention to steady-state analysis and relied on approximation methods with unknown accuracy.

In this paper, we develop a growth model with private information about individual effort (moral hazard) and individual shocks (adverse selection) that is tractable in the sense that optimal allocations do not display a dependence on individual shock histories beyond the current shock realization. Specifically, we consider a dynamic model economy that is populated by a large number of infinitely-lived households who can invest in risk-free physical capital and risky human capital. Human capital investment is risky due to idiosyncratic shocks to the stock of household human capital. Households also make an effort choice that has a utility cost (dis-utility of effort) and affects the probability distribution over idiosyncratic human capital shocks. Specifically, the exogenous shocks follow a Markov process with transition probabilities that depend on effort choices. Households have preferences that allow for a time-additive expected utility representation with a one-period utility function that is additive over consumption and effort as well as logarithmic over consumption. The dis-utility of effort might be subject to idiosyncratic shocks. The production process is represented by a constant-returns-to-scale production function that takes aggregate physical capital and aggregate human capital as input factors.

Constrained optimal allocations are the solution to an infinite-horizon social planner problem with two incentive compatibility constraints. The first constraint ensures that households always have an incentive to choose the individual effort level that is part of the allocation (unobserved effort). The second constraint ensures that households have an incentive to report/reveal their types and history of individual shocks (unobserved shock histories). In addition, a constrained optimal allocation has to satisfy the standard feasibility constraints that represent the production process, capital accumulation, and the aggregate resource constraint.

We provide two necessary conditions for constrained optimal allocations. First, individual consumption has the martingale property. This condition follows directly from the well-known inverse Euler equation and the assumption that utility is logarithmic over consumption. Second, the expected return on human capital investment is equal to the return to physical capital investment for all households with positive human capital investment level. As our proof shows, the second result is quite general and does not depend on our special assumption on preferences.

We use the two necessary conditions to derive a full characterization of constrained optimal allocations. Specifically, we show that constrained optimal allocations are simple in the sense that individual effort choices and individual consumption growth rates are history-independent. Further, constrained optimal allocations are the solutions to a recursive social planner problem that is simple in the sense that exogenous shocks are the only state variables. In other words, the computation of constrained optimal allocations does not require the introduction of additional endogenous state variables (promised utility) and their distribution over individual households.

To streamline the analysis, we develop the main arguments using a basic version of the model with a simple production structure and without aggregate shocks. However, our arguments and proofs can easily be extended to a version with a more general production structure and with aggregate shocks – the details of these extensions are discussed at the end of section 2. Thus, the private-information framework developed in this paper allows for a wide range of applications in macroeconomics that so far have not been studied in the literature because of tractability problems.

Literature. Our paper is related to several strands of the literature. First, there is the large literature on (constrained) optimal allocations in moral hazard economies. See, for example, Hopenhayn and Nicolini (1997) and Pavoni and Violante (2007) for well-known applications to unemployment insurance and welfare programs, and Laffont and Martimort

(2002) for a survey of micro-oriented literature on moral hazard. Our theoretical tractability result echoes the result derived by Holmstrom and Milgrom (1987) and Fudenberg, Holmstrom, Milgrom (1990) for repeated principal-agent problems, but in contrast to these papers we consider a macroeconomic model with an explicit aggregate resources constraint (general equilibrium analysis).

Second, our paper relates to the macroeconomic literature on optimal taxation in economies with private information using the Mirrlees approach (Mirrlees, 1971) – see Stantcheva (2020) for a survey. Our theoretical tractability result resembles the results of Farhi and Werning (2007) and Phelan (2006), who show that constrained optimal allocations in an OLG-model are the solution to a static social planner problem when the social welfare function puts equal weight on all future generations. In other words, they make an assumption about social preferences. In contrast, in this paper we make assumptions about the production structure and about individual preferences to prove tractability. Our analysis and results resemble most closely the work of Farhi and Werning (2012), who analyze the implications of the inverse Euler equation in an economy with private information about idiosyncratic shocks, linear production structure, and a one-period utility function hat is logarithmic over consumption.

Third, our paper is related to the literature on constrained efficient allocations in incomplete-market models (Geanakoplos and Polemarchakis, 1986) that assume an exogenous asset payoff structure and therefore take the lack of certain type of insurance as given. Aiyagari (1995) and Davila et al. (2012) analyze constrained optimal allocations in a neoclassical growth model with idiosyncratic productivity risk and incomplete markets. Krebs (2006) and Toda (2015) discuss the efficiency properties of incomplete-market models with human capital and a production structure similar to the one discussed in this paper, and Gottardi et al. (2015) analyze the optimal level of taxation and debt in this class of models.

2. Model

This section develops the model and defines constrained optimal allocations. Specifically, subsections 2.1 and 2.2 describe the fundamentals of the economy and section 2.3 defines the social planner problem. The framework combines the production structure of the human capital model developed in Krebs (2003,2006) with a dynamic model of unobserved effort choices or shock histories along the lines of Atkeson and Lucas (1992), Fahri and Werning (2012), Golosov et al. (2003), and Phelan and Townsend (1991). The basic framework disregards aggregate shocks and confines attention to a simple production structure. In subsection 2.4 we discuss extensions of the basic framework with aggregate shocks and a more general production structure.

2.1. Preferences and Uncertainty

Time is discrete and open ended. The economy is populated by a unit mass of infinitely-lived households. In each period t, the exogenous part of the individual state of a household is represented by s_t, which captures the effect of idiosyncratic shocks on household preferences and human capital accumulation (see below). We denote by $s^t = (s_0, s_1, \ldots, s_t)$ the history of exogenous shocks up to period t. We assume that the probability of history $s^t = (s_0, s_1, \ldots, s_t)$ depends on effort choices, $e^{t-1} = (e_0, \ldots, e_{t-1})$. More precisely, we assume that the probability of s^t given s_0 depends on effort choices as follows: $\pi_t(s^t|s_0, e^{t-1}) = \pi(s_t|s_{t-1}, e_{t-1}) \times \ldots \times \pi(s_1|s_0, e_0)$, where $\pi(s_t|s_{t-1}, e_{t-1})$ is the probability of state s_t in period t given state s_{t-1} and effort choice e_{t-1} in period $t-1$. In other words, for given effort choices, the shock process is a Markov process with transition probabilities given by $\pi(s'|s, e)$.

Each household is assigned an initial stock of human capital, h_0, and there is a given initial distribution (of households) over initial human capital and shocks, $\pi_0(h_0, s_0)$, that is independent of effort choices. To streamline the exposition, we assume that there are a finite number of realizations, $s_t \in \{1, \ldots, S\}$, and that effort is one-dimensional, $e \in \mathbf{E} \subset \mathbb{R}$, where $\mathbf{E}$ is a subset of the real line.

Households are risk-averse and have identical preferences that allow for a time-additive expected utility representation with one-period utility function that is additive over consumption and effort as well as logarithmic over consumption. Let $\{c_t, e_t|h_0, s_0\}$ stand for the consumption-effort plan of a household of initial type (h_0, s_0). Expected lifetime utility associated with the consumption-effort plan $\{c_t, e_t|h_0, s_0\}$ is then given by

$$U(\{c_t, e_t|h_0, s_0\}, s_0) = \ln c_0(h_0, s_0) - d(e_0(h_0, s_0), s_0) + \sum_{t=1}^{\infty} \sum_{s^t} \beta^t \left[\ln c_t(h_0, s^t) - d(e_t(h_0, s^t), s_t)\right] \pi_t(s^t|s_0, e^{t-1}(h_0, s^{t-1})) \quad (1)$$

where β is the pure discount factor and $d(.)$ is a dis-utility function that is strictly increasing in e. Note that we allow the dis-utility function to depend on the current state, s_t.

2.2. Production, Capital Accumulation, and Resource Constraint

There is one consumption good that is produced using the aggregate production function

$$Y_t = F\left(K_t, H_t\right) \ , \tag{2}$$

where Y_t is aggregate output in period t, K_t is the aggregate stock of physical capital employed in production, and H_t is the aggregate stock of human capital employed in production. We assume that F is a standard neoclassical production function. In particular, F displays constant returns to scale with respect to the two input factors physical capital, K, and human capital, H.

The consumption good can be transformed into the physical capital good one-for-one. In other words, production of the consumption good and production of physical capital employ the same production function, F. The consumption good is perishable and physical capital depreciates at a constant rate, δ_k. Thus, if X_{kt} denotes aggregate investment in physical capital, then the evolution of aggregate physical capital is given by

$$K_{t+1} = (1-\delta_k)K_t + X_{kt} \ . \tag{3}$$

Human capital is produced at the household level. An individual household can transform the consumption good into human capital using a quantity of x_{ht} consumption goods to produce ϕx_{ht} units of human capital. Note that $1/\phi$ is the price of human capital in units of the consumption (physical capital) good. Existing human capital is subject to random shocks, $\eta_t = \eta(s_t)$. The production function and law of motion for household-level human capital, h_t, are described by

$$\begin{aligned} h_{t+1}(h_0, s^t) &= (1+\eta(s_t))h_t(h_0, s^{t-1}) + \phi x_{ht}(h_0, s^t) \\ h_t(h_0, s^t) &\geq 0 \ . \end{aligned} \tag{4}$$

for all household types and histories (h_0, s^t). Note that h_{t+1} is a linear function of x_{ht} and that, as in Krebs (2003,2006), we do not impose a non-negativity constraint on human capital investment, x_{ht}.

The η-term in the human capital accumulation equation (4) represents changes in human capital that are affected by effort choices and do not require (substantial) goods investment. For example, positive human capital growth, $\eta(s) > 0$, can represent learning-by-doing, and in this case $\pi(., e)$ summarizes the effect of work effort on the success of on-the-job learning. Job-to-job transition is a second example of a positive human capital shock, and in this case it is (on-the-job) search effort that determines the likelihood that the positive realization occurs (the search is successful). In contrast, job loss and the associated loss of firm- or occupation-specific human capital is a typical example of a negative realization $\eta(s) < 0$. In this case, $\pi(., e)$ may represent both the effect of work effort on the likelihood of job loss and the effect of search effort during unemployment on the size of human capital loss associated with the job loss.[1]

Aggregate human capital, H, entering the production function (2) is obtained from individual human capital, h, by taking the expectation over shock histories and initial types:

$$\begin{aligned} H_{t+1} &= E[h_{t+1}] \qquad (5) \\ &= \sum_{h_0, s^t} h_{t+1}(h_0, s^t)\pi_t(s^t|s_0, e^{t-1}(h_0, s^{t-1}))\pi_0(h_0, s_0) \; . \end{aligned}$$

In general, we obtain aggregate variables from their individual counterparts as in (5). Taking the expectation over equation (4) yields the aggregate human capital accumulation equation:

$$H_{t+1} = H_t \, + \, E[\eta_t h_t] \, + \, \phi X_{ht} \; , \qquad (6)$$

where $X_{ht} = E[x_{ht}]$ is aggregate investment in human capital. Note that $E[\eta_t h_t] \neq E[\eta_t]E[h_t]$ when e_{t-1} depends on s^{t-1}.

Finally, the aggregate resource constraint in the economy reads:

$$C_t + X_{kt} + X_{ht} \; = \; Y_t \quad . \qquad (7)$$

The resource constraint (7) says that aggregate output produced is equal to the sum of aggregate consumption, aggregate investment in physical capital, and aggregate goods investment in human capital.

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[1] We use $\eta(s_t)$ instead of $\eta(s_{t+1})$ in (3) in order to simplify the formal proofs, a timing choice also made in Krebs (2003) and Stantcheva (2017). However, the current analysis and results apply, mutatis mutandis, if the timing is changed and $\eta(s_{t+1})$ is used in (3). See Stokey and Lucas (1989) for a general discussion of this issue in choice problems under uncertainty.

2.3. Constrained Optimal Allocations

We next define constrained optimal allocations. Consider a social planner who directly chooses an allocation, $\{c_t, e_t, h_{t+1}, K_{t+1}\}$ with $H_{t+1} = E[h_{t+1}]$, subject to the feasibility constraints defined by (2), (3), (4), (7) and additional incentive compatibility constraints. These incentive constraints arise because effort choices are private information (moral hazard) and household types and histories are private information (adverse selection).

To define the incentive constraint associated with private information about household types and histories, we introduce reporting functions (strategies) σ_t that map type-histories, (h_0, s^t), into type-histories $\sigma_t(h_0, s^t)$. Let σ_t^* be the truth-telling reporting function: $\sigma_t^*(h_0, s^t)$ $= (h_0, s^t)$ for all (h_0, s^t). We denote the combination of a consumption-effort plan and reporting strategy by $\{c_t, e_t, \sigma_t | h_0, s_0\}$. Lifetime utility derived from $\{c_t, e_t, \sigma_t | h_0, s_0\}$ by household type (h_0, s_0) is given by

$$
\begin{aligned}
U(\{c_t, e_t, \sigma_t | h_0, s_0\}, s_0) \; = \; & \ln c_0(\sigma_0(h_0, s_0)) - d(e_0(\sigma_0(h_0, s_0)) + \\
& \sum_{t=1}^{\infty} \sum_{s^t} \beta^t \left[\ln c_t(\sigma_t(h_0, s^t)) - d(e_t(\sigma_t(h_0, s^t)), s_t) \right] \pi_t(s^t | s_0, e^{t-1}(\sigma_{t-1}(h_0, s^{t-1}))) \; .
\end{aligned}
\tag{8}
$$

An allocation $\{c_t, e_t, h_{t+1}, K_{t+1}\}$ is incentive compatible if $\{c_t, e_t\}$ satisfies:

$$
\begin{gathered}
\forall \, (h_0, s_0) \, , \, \forall \, \{\hat{e}_t, \hat{\sigma}_t | h_0, s_0\} \; : \\
U_t(\{c_t, e_t, \sigma_t^* | h_0, s_0\}, s_0) \; \geq \; U_t(\{c_t, \hat{e}_t, \hat{\sigma}_t | h_0, s_0\}, s_0) \; .
\end{gathered}
\tag{9}
$$

where σ_t^* is the truth-telling function. Note that in line with the microeconomic literature (Laffont and Martimort, 2002) we assume that the household type, (h_0, s_0), is private information. In contrast, the dynamic public finance literature sometimes assumes that (part of) the household type is known to the social planner (Fahri and Werning, 2012).

Equation (9) formalizes the idea that the social planner cannot observe effort levels and type-histories, and individual households therefore have to have an incentive to adhere to the proposed plan. Private information about individual effort choices (moral hazard) requires that the social planner can only choose consumption-effort allocations, $\{c_t, e_t\}$, that are incentive compatible in the sense that households have an incentive to choose the effort plan for given consumption plan. Private information about individual household types and histories (adverse selection) requires that the social planner can only choose consumption-effort allocations, $\{c_t, e_t\}$, that are incentive compatible in the sense that households have an incentive to report their type and history of shocks truthfully. Private information about both individual effort and individual type-histories requires that both incentive compatibility conditions are satisfied, where all joint deviations in effort and reporting are considered.

We define the constraint set of the social planner problem as the set that satisfies the feasibility constraints and the incentive compatibility constraints:

$$\mathbf{A} \equiv \{\{c_t, e_t, h_{t+1}, K_{t+1}\} | \{c_t, e_t, h_{t+1}, K_{t+1}\} \text{ satisfies } (2), (3), (4), (7), (9)\} . \tag{10}$$

We assume that the social planner's objective function is social welfare defined as the weighted average of the expected lifetime utility of individual households defined in (1), where we use the Pareto weight μ to weigh the importance of households of type (h_0, s_0). For notational simplicity, we assume a finite number of initial types. If $\mu(h_0, s_0) = \pi_0(h_0, s_0)$, then each individual household is assigned equal importance by the social planner.

Definition 2. An *optimal allocation* is the solution to the social planner problem

$$\max_{\{c_t, e_t, h_{t+1}, K_{t+1}\}} \sum_{h_0, s_0} U(\{c_t, e_t | h_0, s_0\}, s_0) \mu(h_0, s_0) \tag{11}$$

$$\text{subject } to : \{c_t, e_t, h_{t+1}, K_{t+1}\} \in \mathbf{A}$$

where the constraint set $\mathbf{A}$ is defined in equation (10).

As in Golosev et al. (2003) and Farhi and Werning (2012), we assume that physical capital production is not subject to (idiosyncratic) risk and our definition of (optimal) allocations therefore only refers to the aggregate physical capital stock, K. In contrast, human capital is produced at the household level and the allocation of human capital across households is therefore specified as part of an (optimal) allocation.

2.4. Extensions

There are four main extensions of the basic framework that can be incorporated without sacrificing the tractability of the model. Specifically, the main characterization results (propositions 1-3) still hold, mutatis mutandis, and proofs of the these results are similar to the ones given in this paper.

First, we can introduce additional sources of idiosyncratic investment and production risk. Specifically, the productivity of human capital investment can be subject to idiosyncratic shocks, $\phi = \phi(s_t)$. Further, the productivity of human capital can be subject to idiosyncratic risk, which amounts to replacing H in the production function (2) by $E[z(s_t)h_t]$.

Second, economic fundamentals may depend on aggregate shock, S_t. Specifically, the stochastic process of exogenous shocks can be a Markov process for given effort choices with transition probabilities $\pi(s_{t+1}, S_{t+1}|s_t, S_t, e_t)$, where certain restrictions should be placed on π to ensure that individual effort choices do not affect the probability of the aggregate shock. In this case, the main characterization results for optimal allocations still hold in the sense that effort choices and individual consumption growth rates are independent of individual histories and type (proposition 3), but now they depend on (s_t, S_t).

Third, equation (4) representing the production of human capital can also be generalized. As in Krebs (2003,2006) and Stantcheva (2017), equation (4) assumes that human capital production only uses goods. In contrast, Heckman, Lochner, and Taber (1998) and Huggett, Ventura, and Yaron (2011) focus on the time investment in human capital. Clearly, in most cases human capital investment uses both goods and time. The tractability result derived in this paper also holds for the case in which both goods and time are used to produce human capital as long as there is constant-returns-to-scale. Specifically, we can introduce a time cost of human capital production by replacing the term ϕx_{ht} in (4) by $\phi \left(h_t l_t\right)^\rho x_{xt}^{1-\rho}$, where l_t denotes the time spend in human capital production. If there is a fixed amount of time that is allocated between producing human capital, l_t, and working, $1 - l_t$, it is straightforward to show that this human capital production function gives rise to a human capital accumulation equation (4) that is still linear in x_{ht} after substituting out the optimal choice of l_t.

Finally, as in Jones and Manuelli (1990) and Rebelo (1991), the aggregate production function (2) displays constant-returns-to-scale with respect to production factors that can be accumulated without bounds, a property that is well-known to generate endogenous growth. The main results of this paper still hold if (2) is replaced by a production function with diminishing returns or, equivalently, a production function with constant-returns-to-scale and a third (fixed) factor of production (land). However, in this case we have an explicit time-dependence of individual and aggregate variables, and convergence towards a steady state instead of unbounded growth under certain conditions.

3. Results

This section states and discusses the theoretical results. Subsections 3.1 and 3.2 provide two characterization results for constrained optimal allocations: Expected returns are equalized across investment opportunities (proposition 1) and individual consumption has the martingale property (proposition 2). Subsection 3.3 provides a full characterization of constrained optimal allocations and shows that they are simple (proposition 3). Proofs of the propositions are collected in the Appendix.

3.1 Production Efficiency

Consider an allocation $\{c_t, e_t, h_{t+1}, K_{t+1}\}$. In economies with complete information, production efficiency requires that expected returns on alternative investment opportunities are equalized if investment levels are positive.[2] In the model considered in this paper, this equalization-of-returns condition reads:

$$\phi\, F_h(\tilde{K}_{t+1}) \;+\; \sum_{s_{t+1}} \eta(s_{t+1})\pi(s_{t+1}|s_t, e_t(h_0, s^t)) \;=\; F_k(\tilde{K}_{t+1}) - \delta_k \qquad (12)$$

Proposition 1 below shows that the optimality condition (12) also characterize optimal allocations in our private information economy for all households (h_0, s^t) with positive human capital, $h_t(h_0, s^t) > 0$. Clearly, the efficiency condition (12) does not have to hold for histories with $h_t(h_0, s^t) = 0$. However, even for those histories an inequality version of (12) holds: Expected human capital returns cannot exceed the return to physical capital investment. In addition, a standard argument shows that the optimal $\tilde{K}_t$ is independent of t since production displays constant returns to scale with respect to H and K, and these two factors of production can be adjusted at no cost. Thus, we have the following result:

[2] More precisely, if a capital allocation maximizes aggregate output net of depreciation, then the (expected) returns on physical capital investment and human capital investment are equalized. Further, the capital-to-labor ratio that maximizes the expected total investment return for given effort level is determined by the equality-of-returns condition.

Proposition 1. A constrained optimal allocation exists. The optimal aggregate capital-to-labor ratio is constant over time: $\tilde{K}_t = \tilde{K}$ for all periods $t = 1, \ldots$. Further, for all household types and histories, (h_0, s^t), the expected return on human capital investment cannot exceed the return on physical capital investment:

$$\phi F_h(\tilde{K}) + \sum_{s_{t+1}} \eta(s_{t+1}) \pi(s_{t+1}|s_t, e_t(h_0, s^t)) \leq F_k(\tilde{K}) - \delta_k , \tag{13}$$

where (13) holds with equality for all (h_0, s^t) with positive human capital, $h_t(h_0, s^t) > 0$.

Proof: See appendix.

Proposition 1 states that, under certain conditions, a standard production efficiency condition has to hold even if there is private information. In this sense the result resembles the original result by Diamond and Mirrlees (1971). The optimality of the equality-of-return condition (12), respectively (13), was first shown by Da Costa and Maestri (2007) in a one-period model of human capital investment with private information about type (adverse selection).

The proof of proposition 1 is quite general and does not hinge on the linearity of individual human capital investment opportunities. The crucial assumption is that human capital investment is observable, but beyond this informational assumption not much is needed for the proof. Indeed, the proof conducted in the Appendix shows that the result holds for any production function (2) and any human capital accumulation equation of the type $h_{t+1} = g(h_t, x_{ht}, l_t, s_t)$ as long as financial investment (borrowing and lending) and human capital investment (labor income) are observable, where l_t is the time spent in human capital production. For the general case the human capital return has to be defined as $r_{h,t+1} = g_{x_{ht}}((1 - l_{t+1})F_{h,t+1} + g_{h,t+1}/g_{x_{h,t+1}}) - 1$.

One direct implication of proposition 1 is that effort choices are the same for households with positive human capital: $e_t(h_0, s^t) = e^*$ for all (h_0, s^t) with $h_{t+1}(h_0, s^t) > 0$. This follows since different effort choices lead to different values of $\sum_{s_{t+1}} \eta(s_{t+1}) \pi(s_{t+1}|s_t, e_t(h_0, s^t))$ if we place a corresponding joint restriction on η and π. However, $h_{t+1}(h_0, s^t) = 0$ cannot be ruled out as part of an optimal allocation and we therefore need a further characterization result (proposition 2) to prove the simplicity of constrained optimal allocations (proposition 3).

3.2. Inverse Euler Equation

We continue to consider allocations $\{c_t, e_t, h_{t+1}, K_{t+1}\}$. Equation (13) defines one set of necessary conditions for optimal allocations. Another set of necessary conditions is provided by the inverse Euler equation (Golosov et al. 2003, Rogerson, 1985a), which holds in any private information economy with a saving technology including models with human capital investment (Stantcheva, 2017). In the current framework, this inverse Euler equation reads

$$c_t(h_0, s^t) \;=\; \left[\beta\left(1 + r(\tilde{K})\right)\right]^{-1} \sum_{s_{t+1}} c_{t+1}(h_0, s^{t+1})\pi(s_{t+1}|s_t, e_t(h_0, s^t)) \;, \tag{14}$$

where $r(\tilde{K}) = F_k(\tilde{K}) - \delta_k$ denote the (risk-free) rate on physical capital investment. Equation (14) says that expected consumption growth is equal to $\beta(1+r)$ for all (h_0, s^t). In other words, optimal individual consumption has the martingale property – see Ljungqvist and Sargent (2018) for a discussion of the martingale property in economics. The optimal individual consumption process follows a sub-martingale if $\beta(1+r) > 1$, a martingale if $\beta(1+r) = 1$, and a super-martingale if $\beta(1+r) < 1$.

A direct implication of the martingale property (14) is that optimal individual consumption can be represented as

$$c_{t+1}(h_0, s^{t+1}) \;=\; \beta\left(1 + r(\tilde{K}) + \epsilon_{t+1}(h_0, s^{t+1})\right) c_t(h_0, s^t) \tag{15}$$

where ϵ is a random variable that represents risk in individual consumption growth in the sense that its conditional mean is zero:

$$\sum_{s_{t+1}} \epsilon_{t+1}(h_0, s^{t+1})\pi(s_{t+1}|s_t, e_t(h_0, s^t)) \;=\; 0 \;. \tag{16}$$

Clearly, the choice of a consumption-effort allocation, $\{c_t, e_t\}$, is equivalent to the choice of a risk-effort allocation, $\{e_t, \epsilon_{t+1}\}$, together with a choice of initial consumption function, $c_0 = c_0(h_0, s_0)$.

The preceding discussion is summarized in the following proposition:

Proposition 2. Any constrained optimal allocation satisfies (15). In other words, optimal individual consumption growth is given by $\beta(1 + r(\tilde{K}) + \epsilon_{t+1})$, where $E_e[\epsilon_{t+1}|h_0, s^t] = 0$ for all (h_0, s^t).

Proof: See appendix.

Note that taking the expectations over (h_0, s^t) in (15) shows that optimal aggregate consumption follows $C_{t+1} = \beta(1+r(\tilde{K}))C_t$. Thus, for given $\tilde{K}$ and C_0, the optimal aggregate consumption path is pinned down by the inverse Euler equation in the current setting.

3.3. Full Characterization

Proposition 2 establishes that in our search for constrained optimal consumption-effort allocations, $\{c_t, e_t\}$, we can restrict attention to risk-effort allocations, $\{\epsilon_{t+1}, e_t\}$, and a distribution or initial consumption levels, $c_0(.)$, where $\epsilon_{t+1}(.)$ satisfies (16). Consider an optimal $\{c_0, \epsilon_{t+1}, e_t\}$ and assume that, in addition to (15), consumption risk and effort choices are independent of household type and history: $e_t(h_0, s^t) = e(s_t)$ and $\epsilon_{t+1}(h_0, s^t, s_{t+1}) = \epsilon(s_t, s_{t+1})$. In this case, using the consumption representation (15) and the definition of lifetime utility, respectively lifetime continuation utility, simple algebra shows that the infinite-horizon social planner problem (11) is equivalent to the simple social planner problem

$$\max_{e,\epsilon,\tilde{K}} \left\{ \sum_s V(s, e(s), \epsilon(s, .), \tilde{K})\, \mu(s) \right\}$$

$$\text{subject:} \tag{17}$$

$$\forall\, s \;:\; r(\tilde{K}) \;=\; \phi\, F_h(\tilde{K}) \;+\; \sum_{s'} \eta(s')\pi(s'|s, e(s))$$

$$\forall\, s \;:\; \sum_{s'} \epsilon(s, s')\pi(s'|s, e(s)) \;=\; 0$$

$$\forall\, s\,,\, \sigma(s)\,,\, \hat{e}(\sigma(s)) \;:\; V(s, e(s), \epsilon(s, .), \tilde{K}) \;\geq\; V(s, \hat{e}(\sigma(s)), \epsilon(\sigma(s), .), \tilde{K})$$

where $\mu(s) = \sum_h \mu(h, s)$. Further, the intensive-form value function, V, solves the simple recursive equation:

$$V(s) \;=\; -d(e(s), s) \;+\; B(\beta) \;+\; \frac{\beta}{1-\beta} \sum_{s'} \ln(1 + r(\tilde{K}), \epsilon(s, s'))\pi(s'|s, e(s)) \tag{18}$$

$$+\,\beta \sum_{s'} V(s', e(s'), \epsilon(s, s'), \tilde{K})\pi(s'|s, e(s))$$

with $B(\beta) = \ln(1-\beta) + \frac{\beta}{1-\beta}\ln\beta$. The recursive equation (18) is simple because only the exogenous state/shock, s, enters into the equation; no additional endogenous state (promised utility) is needed to obtain the solution. Note that the objective function (social welfare) in the maximization problem (17) allows for a recursive representation because it is defined as the sum of recursively defined functions. In this sense the social planner problem defined by (17) and (18) is a recursive problem even though it appears to be different from the standard recursive problems in macroeconomics (Stokey and Lucas, 1989). Note further that lifetime utility of a household of initial type (s_0, h_0) is given by:

$$U(\{c_t, e_t|h_0, s_0\}, s_0) \;=\; \frac{1}{1-\beta}\ln c_0(h_0, s_0) \;+\; V(s_0, e(s_0), \epsilon(s_0, .), \tilde{K})\;. \tag{19}$$

The following proposition states that the focus on simple allocations in our search for optimal allocations is justified:

Proposition 3. Optimal allocations are simple. Specifically, let the triple $(e^*, \epsilon^*(.), \tilde{K}^*)$ be the solution to the static social planner problem (17), where the value function, V, is defined by the simple recursive equation (18). Then the optimal allocation is given by:

$$\begin{aligned}
e_t(h_0, s^t) &= e^*(s_t) \\
\epsilon_{t+1}(h_0, s^{t+1}) &= \epsilon^*(s_t, s_{t+1}) \\
\tilde{K}_{t+1} &= \tilde{K}^* \\
c_{t+1}(h_0, s^{t+1}) &= \beta\left(1 + r(\tilde{K}^*) + \epsilon^*(s_t, s_{t+1})\right) c_t(h_0, s^t) \\
c_0(h_0, s_0) &= (1-\beta)\left(1 + r(\tilde{K}_0)\right)(K_0 + H_0/\phi)\,\frac{\mu(h_0, s_0)}{\pi_0(h_0, s_0)} \\
C_{t+1} &= \beta(1 + r(\tilde{K}^*))C_t \\
K_{t+1} &= \beta(1 + r(\tilde{K}^*))K_t \\
H_{t+1} &= \beta(1 + r(\tilde{K}^*))H_t\;.
\end{aligned} \tag{20}$$

Further, lifetime utility of individual households is given by (19).

Proof: See appendix.

Note that even though the optimal aggregate level of human capital investment, X_{ht}, is uniquely determined for all t, the optimal level of individual human capital investment is indeterminate since the optimal effort choice, $e^*(s)$, is common across households with the same s.

Consider the case in which the shock process is i.i.d. for given effort choice and there are no dis-utility shocks. In other words, neither $\pi(s_{t+1}|s_t, e_t)$ nor $d(e_t, s_t)$ depend on s_t. In this case, lifetime utility, $U_t(\{c_{t+n}, e_{t+n}|h_0, s_0\})$, does not explicitly depend on s_0. Proposition 3 implies that the value function, V, also does not depend on s and reads:

$$V(e, \epsilon(.), \tilde{K}) \;=\; \frac{1}{1-\beta}\left[-d(e) \,+\, g(\beta) \,+\, \frac{\beta}{1-\beta}\sum_{s'} ln\left(1 + r(\tilde{K}) + \epsilon(s')\right)\pi(s'|e)\right] \tag{21}$$

Further, in this case the social planner problem (17) reduces to the following constrained maximization problem:

$$\begin{aligned} &\max_{e,\epsilon,\tilde{K}} \left[-d(e) \,+\, \frac{\beta}{1-\beta}\sum_{s'} ln\left(1 + r(\tilde{K}) + \epsilon(s')\right)\pi(s'|e)\right] \\ &\text{subject to:} \\ &r(\tilde{K}) \;=\; \phi\, F_h(\tilde{K}) \,+\, \sum_{s'}\eta(s)\pi(s'|e) \\ &\sum_{s'}\epsilon(s')\pi(s'|e) \;=\; 0 \\ &\forall\, \hat{e}: \quad -d(e) \,+\, \frac{\beta}{1-\beta}\sum_{s'} ln\left(1 + r(\tilde{K}) + \epsilon(s')\right)\pi(s'|e) \\ &\qquad \geq\, -d(\hat{e}) \,+\, \frac{\beta}{1-\beta}\sum_{s'} ln\left(1 + r(\tilde{K}) + \epsilon(s')\right)\pi(s'|\hat{e}) \end{aligned} \tag{22}$$

In sum, we have the following corollary:

Corollary Suppose that the shock process is i.i.d. for given effort choice and there are no dis-utility shocks. Then the social planner problem (17) reduces to (22).

The maximization problem (22) is the choice problem of a social planner who chooses effort level, e, consumption risk, ϵ, and a capital-to-labor ratio, $\tilde{K}$, so as to maximize welfare defined by the expected utility of households with log-utility function and consumption given by $\ln(1 + r(\tilde{K}) + \epsilon')$ subject to three constraints. The first constraint states that the return to financial capital investment is equal to the expected return to human capital investment, where the social planner can affect returns through the choice of the capital-to-labor ratio and the mean level of human capital shocks (effort). The second constraint says that ϵ is a variable representing risk and therefore has a fixed mean, which is normalized to zero.

The final constraint is the incentive compatibility constraint that ensures that individual households will choose the prescribed effort choice.

A number of comments regarding proposition 3 are in order. First, proposition 3 implies that the cross-sectional distribution of consumption spreads out over time – the well-known immiseration result of Atkeson and Lucas (1992). If we introduce an OLG-structure with stochastic death of households (Contantinides and Duffie, 1996) and a social welfare function that puts weight on future generations (Farhi and Werning, 2007, and Phelan, 2006), we can generate a stationary cross-sectional distribution of consumption while still keeping the tractability of the model. However, the cross-sectional distributions of consumption and wealth still exhibit fat tails and obey the double power law (Toda, 2014).

Second, proposition 3 rules out that households enter an absorbing state in which consumption is constant and effort is zero – the "retirement" state in the language of Sannikov (2008). In the current model, retirement at low levels of consumption does not occur because utility is not bounded from below. In addition, retirement at high levels of consumption is not optimal because preferences are consistent with balanced growth so that the (relative) cost of providing incentives to induce positive effort choices are independent of the level of consumption, that is, income and substitution effect of increases in income/wealth cancel each other out.

Finally, for the specification with i.i.d. shocks, with some additional assumptions we can replace the inequality constraints in the maximization problem (22) by the first-order conditions to characterize the optimal effort choice of individual households for given level of consumption risk, which read:

$$d'(e) \;=\; \frac{\beta}{1-\beta}\sum_{s'} ln\left(1+r(\tilde{K})+\epsilon(s')\right)\frac{\partial\pi}{\partial e}(s'|e) \qquad (23)$$

Note that in our setting we can use well-known results for one-period moral hazard problems (Rogerson, 1985b) to ensure that the first-order condition approach is appropriate because of proposition 3. In contrast, for general repeated moral hazard economies the first-order conditions might not be sufficient since the product of two concave (probability) functions is not necessarily concave, and there are no results for general repeated moral hazard problems in the literature. Abaraham, Koehne, and Pavoni (2011) provide conditions for a two-period moral hazard problem that ensure necessity and sufficiency of first-order conditions.

Appendix

Proof of Proposition 1.

Clearly, a straightforward approach to deriving the necessity of condition (12), respectively (13), is to write down the Lagrangian associated with the social planner problem and then to take first-order conditions. However, the existence of a vector of Lagrange multipliers requires additional conditions that might not be satisfied.[3] We therefore use a direct approach that does not require any assumptions on the primitives beyond the once already made in the paper.

For notational ease, we consider the case in which shock process is i.i.d. for given effort choice and disregard utility shocks. We further suppress the dependence of plans on h_0. To prove the claim, suppose not, that is, for the optimal allocation $\{c_t, e_t, k_t, h_t\}$ there exists a $\bar{t}$ and $\bar{s}^{\bar{t}}$ with $h_t(\bar{s}^{\bar{t}}) > 0$ and (12) is not satisfied:

$$\phi\, F_h(\tilde{K}_{\bar{t}+1}) \,+\, \sum_{s_{\bar{t}+1}} \eta(s_{\bar{t}+1})\pi(s_{\bar{t}+1}|e_t(\bar{s}^{\bar{t}})) \quad > \quad F_k(\tilde{K}_{\bar{t}+1}) - \delta_k \ . \tag{A1}$$

Inequality (A1) states that the expected value of human capital returns (the left-hand-side of A2) exceeds the risk-free return on physical capital investment (the right-hand-side of A2). The proof by contradiction for the reversed case is, mutatis mutandis, the same.

Consider an alternative allocation $\{\hat{c}_t, e_t, \hat{k}_t, \hat{h}_t\}$ with identical $\{e_t\}$ and a $\{\hat{c}_t, \hat{k}_t, \hat{h}_t\}$ that only differs from $\{c_t, k_t, h_t\}$ at history $\bar{s}^{\bar{t}}$ and for all $s_{\bar{t}+1}$ subsequent to $\bar{s}^{\bar{t}}$. More specifically, we define

$$\begin{aligned} \hat{h}_{\bar{t}+1}(\bar{s}^{\bar{t}}) &= h_{\bar{t}+1}(\bar{s}^{\bar{t}}) \,+\, (1+\eta(s_t))h_t + \phi\,(x_{ht} + \Delta x) \\ \hat{k}_{\bar{t}+1}(\bar{s}^{\bar{t}}) &= k_{\bar{t}+1}(\bar{s}^{\bar{t}}) - \Delta x \\ \forall s_{\bar{t}+1} \ : \ \hat{c}_{\bar{t}+1}(\bar{s}^{\bar{t}}, s_{\bar{t}+1}) &= c_{\bar{t}+1}(\bar{s}^{\bar{t}}, s_{\bar{t}+1}) + \Delta c(s_{\bar{t}+1}) \ , \end{aligned} \tag{A2}$$

where the changes $\Delta x > 0$ and $\Delta c(s_{\bar{t}+1}) > 0$ are strictly positive real numbers. In words: in period $\bar{t}$, the alternative allocation increases human capital investment by Δx and reduces physical capital investment by Δx for households of type $\bar{s}^{\bar{t}}$, and in period $\bar{t}+1$ it increases consumption for these households in all possible states. Clearly, this allocation strictly increases social welfare. We now show that such a strictly positive vector $(\Delta x, \vec{\Delta c})$ exists

[3] See Rustichini (1998) for a general treatment of the question of the existence of a Lagrange vector in infinite-dimensional optimization problems with incentive constraints.

so that $\{\hat{c}_t, e_t, \hat{k}_t, \hat{h}_t\}$ satisfies the aggregate resource constraint and the incentive constraint, which contradicts the claim that $\{c_t, e_t, k_t, h_t\}$ is an optimal allocation. The idea of the proof is to show that the investment change increases available resources in $\bar{t}+1$ for small enough Δx and that the additional resources can be used to increase consumption in each state $s_{\bar{t}+1}$ without affecting the incentive constraint.

Since F is continuously differentiable the increase in human capital investment in period $\bar{t}$ by Δx increases production in period $\bar{t}+1$ by

$$\phi F_{h,\bar{t}+1}\Delta x \;+\; \epsilon_1(\Delta x) \tag{A3}$$

with $\lim_{\Delta x\to 0}\frac{\epsilon_1(\Delta x)}{\Delta x}=0$. To reverse the increase in human capital investment in period $\bar{t}$, in the alternative allocation investment in human capital in period $\bar{t}+1$ is reduced by $\Delta x'(s_{\bar{t}+1})$. Since we require $\hat{h}_{\bar{t}+2}=h_{\bar{t}+2}$, the two investment changes Δx and $\Delta x'$ need to satisfy

$$\Delta x'(s_{\bar{t}+1}) \;=\; (1+\eta(s_{\bar{t}+1}))\Delta \tag{A4}$$

Finally, the reduction in investment in physical capital in period $\bar{t}$ by Δx reduces output by $(F_{k,\bar{t}+1}-\delta_k)\,\Delta x \,+\, \epsilon_2(\Delta x)$ and the increase in physical capital investment in period $\bar{t}+1$ by Δx necessary to achieve $\hat{k}_{\bar{t}+2}(\bar{s}^{\bar{t}}, s_{\bar{t}+1}) = k_{\bar{t}+2}(\bar{s}^{\bar{t}}, s_{\bar{t}+1})$ reduces available resources in period $\bar{t}+1$ by $\Delta x+\epsilon_3(\Delta x)$, where $\lim_{\Delta x\to 0}\frac{\epsilon_2(\Delta x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\epsilon_3(\Delta x)}{\Delta x}=0$.

In sum, for the alternative allocation $\{\hat{c}_t, e_t, \hat{k}_t, \hat{h}_t\}$ the additional resources available for consumption in period $\bar{t}+1$ for households of type $\bar{s}^{\bar{t}}$ are

$$\begin{aligned}\Delta\omega \;&=\; \phi F_{h,\bar{t}+1}\,\Delta x \\ &+\; \left(1+\sum_{s_{\bar{t}+1}}\eta(s_{\bar{t}+1})\pi(s_{\bar{t}+1}|e_{\bar{t}}(\bar{s}^{\bar{t}})\right)\Delta x \\ &-\; (1+F_{1,\bar{t}+1}-\delta_k)\,\Delta x \;+\; \epsilon(\Delta x)\end{aligned} \tag{A5}$$

with $\lim_{\Delta x\to 0}\frac{\epsilon(\Delta x)}{\Delta x}=0$. Using the assumption that expected human capital returns exceed the financial returns, we conclude that for small enough Δx we have $\Delta\omega>0$.

Take a strictly positive real number Δu and define $\Delta c(s_{\bar{t}+1})$, for each $s_{\bar{t}+1}$, as the solution to

$$\ln\Big(c_{\bar{t}+1}(\bar{s}^{\bar{t}})+\Delta c(s_{\bar{t}+1})\Big) \;=\; \ln\Big(c_{\bar{t}+1}(\bar{s}^{\bar{t}})\Big) \;+\; \Delta u \tag{A6}$$

Since the logarithmic function is continuous and strictly increasing in c we can always find positive real numbers $\Delta c(s_{\bar{t}+1})$ so that (A6) holds for given Δu. Further, continuous differentiability of the logarithmic function implies for sufficiently small Δu that the solution

$\Delta\vec{c}$ to (A6) satisfies $\sum_{s_{\bar{t}+1}} \Delta c(s_{\bar{t}+1})\pi(s_{\bar{t}+1}|e_t(\bar{s}^{\bar{t}})) = \Delta\omega$. Thus, the alternative allocation $\{\hat{c}_t, e_t, \hat{k}_t, \hat{h}_t\}$ satisfies the aggregate resource constraint. It also satisfies the incentive constraint since

$$\begin{aligned} \sum_{s_{\bar{t}+1}} \ln\left(\hat{c}_{\bar{t}+1}(\bar{s}^{\bar{t}})\right)\pi(s_{\bar{t}+1}|e_t(\bar{s}^{\bar{t}})) &= \sum_{s_{\bar{t}+1}} \ln\left(c_{\bar{t}+1}(\bar{s}^{\bar{t}}) + \Delta c(s_{\bar{t}+1})\right)\pi(s_{\bar{t}+1}|e_t(\bar{s}^{\bar{t}})) \\ & \sum_{s_{\bar{t}+1}} \ln\left(c_{\bar{t}+1}(\bar{s}^{\bar{t}})\pi(s_{\bar{t}+1}|e_t(\bar{s}^{\bar{t}}))\right) + \Delta u \end{aligned} \quad \text{(A7)}$$

for any probability distribution π over states $s_{\bar{t}+1}$. This completes the proof of proposition 1.

Proof of Proposition 2.

For economies without human capital investment, Fahri and Werning (2012), Golosov et al. (2003) or Rogerson (1985a) show that any constrained optimal allocation with $X_{kt} > 0$ has to satisfy an inverse Euler equation, which in our setting boils down to (14) since c_t the inverse of the marginal utility of consumption. The proof only requires that aggregate consumption can be shifted across periods through adjustments in physical capital investment, which means that the inverse Euler equation (14) is also a necessary condition for optimal allocation when human capital is a choice variable. Stantcheva (2017) contains an explicit proof of the necessity of the Euler equation in economies with human capital investment. Note that equation (14) has to hold for all types (h_0, s_0) and all histories s^t, including initial states and histories with $h_t(h_0, s^t) = 0$.

A direct implication of the martingale property (14) is that optimal individual consumption can be represented as

$$c_{t+1}(h_0, s^{t+1}) = \beta\left(1 + r(\tilde{K}(e^*)) + \epsilon_{t+1}(h_0, , s^{t+1})\right) c_t(h_0, s^t) \quad \text{(A8)}$$

where ϵ is a random variable that represents risk in individual consumption growth and has to satisfy

$$\sum_{s_{t+1}} \epsilon_{t+1}(h_0, s^{t+1})\pi(s_{t+1}|s_t, e_t(h_0, s^t)) = 0 \quad \text{(A9)}$$

This proves the proposition. Note that (A9) only requires ϵ_{t+1} to have a conditional mean of zero, $E_e[\epsilon_{t+1}|h_0, s^t] = 0$, but still leaves open the possibility that the distribution of ϵ_{t+1} varies with time. In other words, $\{\epsilon_t\}$ can be any sequence of mean-zero random variables.

Proof of Proposition 3.

According to the Weierstrass Theorem it suffices to show that the objective function in the maximization problem (11) is upper semi-continuous and the constraint set is compact. Using a variant of the arguments made in Becker and Boyd (1997), a straightforward argument shows that both properties hold if we choose the product topology to define the underlying metric space.

For notational ease, we consider the case in which the shock process is i.i.d. for given effort choice and disregard dis-utility shocks. Propositions 1 and 2 imply that we can write lifetime utility as:

$$U(\{c_t, e_t | h_0, s_0\}) \; = \; \ln c_0(h_0, s_0) \; + \; \tilde{U}_0(\{\epsilon_t, e_t | h_0, s_0\}) \; . \tag{A10}$$

with $\tilde{U}_0$ given by:

$$\tilde{U}_0(\{\epsilon_t, e_t | h_0, s_0\}) \; = \; -d(e_0(h_0, s_0)) + \tag{A11}$$
$$+ \sum_{t=1}^{\infty} \sum_{s^t} \beta^t \left[\ln(1 + r(\tilde{K}) + \epsilon_t(h_0, s^t)) - d(e_t(h_0, s^t)) \right] \pi_t(s^t | e^{t-1}(h_0, s^{t-1})) \; .$$

Denote the continuation plan for type h_0 at history s^t by $\{\epsilon_{t+1+n}, e_{t+n} | h_0, s^t\}$ and denote the corresponding continuation value by $\tilde{U}_t$. Note that $\tilde{U}_t$ satisfies the recursive equation

$$\tilde{U}_t(\{\epsilon_{t+n}, e_{t+n} | h_0, s^t\}) \; = \; -d(e_t(h_0, s^t)) + g(\beta) + \frac{\beta}{1-\beta} \sum_{s_{t+1}} \ln(1 + r(\tilde{K}) + \epsilon_{t+1}(h_0, s^{t+1})) \pi(s_{t+1} | e^t(h_0, s^t)) \tag{A12}$$
$$\beta \sum_{s_{t+1}} \tilde{U}_{t+1}(\{\epsilon_{t+1+n}, e_{t+1+n} | h_0, s^{t+1}\}) \pi(s_{t+1} | e^t(h_0, s^t)) \; .$$

The incentive constraint (9) implies that for all (h_0, s^t) and $(\hat{h}_0, \hat{s}^t)$

$$\tilde{U}_t(\{\epsilon_{t+n}, e_{t+n} | h_0, s^t\}) \; \geq \; \tilde{U}_t(\{\epsilon_{t+n}, e_{t+n} | \hat{h}_0, \hat{s}^t\}) \; . \tag{A13}$$

Inequality (A13) states that each household (h_0, s^t) has an incentive to report his initial type and history truthfully. Clearly, (A13) holds if and only if

$$\tilde{U}_t(\{\epsilon_{t+n}, e_{t+n} | h_0, s^t\}) \; = \; \tilde{U}_t(\{\epsilon_{t+n}, e_{t+n} | \hat{h}_0, \hat{s}^t\}) \tag{A14}$$

for all (h_0, s^t) and $(\hat{h}_0, \hat{s}^t)$. Using the recursive formula (A12), the equality conditions (A14) reduce to

$$-d(e_t(h_0, s^t)) \; + \; \frac{\beta}{1-\beta} \sum_{s_{t+1}} \ln(1 + r(\tilde{K}) + \epsilon_{t+1}(h_0, s^t, s_{t+1})) \pi(s_{t+1} | e^t(h_0, s^t)) \; = \tag{A15}$$

$$-d(e_t(\hat{h}_0, \hat{s}^t)) \; + \; \frac{\beta}{1-\beta} \sum_{s_{t+1}} \ln(1 + r(\tilde{K}) + \epsilon_{t+1}(\hat{h}_0, \hat{s}^t, s_{t+1}))\pi(s_{t+1}|e^t(\hat{h}_0, \hat{s}^t)) \; .$$

In other words, any two next period's risk-effort pairs, $(e, \epsilon(.))$ and $(\hat{e}, \hat{\epsilon}(.))$, that are part of an optimal allocation have to satisfy:

$$-d(e) + \frac{\beta}{1-\beta} \sum_{s'} \ln(1+r(\tilde{K})+\epsilon(s'))\pi(s'|e) \; = \; -d(\hat{e}) + \frac{\beta}{1-\beta} \sum_{s'} \ln(1+r(\tilde{K})+\hat{\epsilon}(s'))\pi(s'|\hat{e}) \; . \tag{A16}$$

The incentive constraint (9) also implies that for all (h_0, s^t) we have

$$\tilde{U}_t(\{\epsilon_{t+n}, e_{t+n}|h_0, s^t\}) \; \geq \; \tilde{U}_t(\{\epsilon_{t+n}, \hat{e}_{t+n}|h_0, s^t\}) \; . \tag{A17}$$

for all alternative effort continuation plans $\{\hat{e}_{t+n}|h_0, s^t\}$. Using the recursive representation of $\tilde{U}$ given by (A12), the inequality (A17) implies that any next period's risk-effort pair, $(e, \epsilon(.))$, that is part of an optimal allocation has to satisfy:

$$-d(e) + \frac{\beta}{1-\beta} \sum_{s'} \ln(1+r(\tilde{K})+\epsilon(s'))\pi(s'|e) \geq \; -d(\hat{e}) + \frac{\beta}{1-\beta} \sum_{s'} \ln(1+r(\tilde{K})+\epsilon(s'))\pi(s'|\hat{e}) \; . \tag{A18}$$

for all alternative effort choices, $\hat{e}$. It is straightforward to show that (A16) and (A18) permit at most one solution, e^*, since d is strictly increasing in e and ln is a concave function. Note that (A16) and (A18) only pin down a unique value of e for given $\tilde{K}$, which in turn depends on the e through the equality-of-return condition. This proves that effort choices and consumption risk are independent of shock histories. Simple algebra then shows that the social planner problem (11) reduces to the social planner problem (17). This proves proposition 3.

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