Optimal Social Insurance and Rising Labor Market Risk*

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Abstract

This paper analyzes the optimal response of the social insurance system to a rise in labor market risk. To this end, the paper develops a tractable macroeconomic model with risk-free physical capital, risky human capital (labor market risk) and unobservable effort choice affecting the distribution of human capital shocks (moral hazard). The paper shows that constrained optimal allocations are simple in the sense that they can be found by solving a static social planner problem. Constrained optimal allocations are the equilibrium allocations of a market economy in which the government uses transfer payments that are linear in household wealth/income to provide optimal insurance and flat-rate income taxes/subsidies to provide optimal investment incentives. The paper shows that an increase in labor market (human capital) risk increases social welfare if the government adjusts the tax-and-transfer system optimally. A quantitative application to the observed rise in job displacement risk in the US shows that the welfare cost of not adjusting the social insurance system optimally can be substantial.

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1. Introduction

A large empirical literature has documented that earnings inequality has been trending upwards in the US and many other countries. There is also considerable evidence that part of this rise in inequality has been driven by a secular increase in the volatility of individual earnings – the labor market has become a more risky place. These stylized facts have generated an influential macroeconomic literature on the causes and consequences of rising inequality and rising uncertainty. For example, Ljungqvist and Sargent (1998) use an increase in economic turbulence in the labor market to explain the secular rise in European unemployment starting in the mid 1970s and motivate their follow-up work in Ljungqvist and Sargent (2008) by the following quote taken from Heckman (2003, p. 30-31):

"A growing body of evidence points to the fact that the world economy is more variable and less predictable today than it was 30 years ago ... [there is] more variability and unpredictability in economic life."

In this paper, we ask two questions. First, what is the welfare effect of the observed rise in labor market risk when the government reacts by adjusting the social insurance system optimally? Second, what is the welfare cost of not adjusting the social insurance system?

We address these two questions using a tractable macroeconomic model with risky human capital and moral hazard. Specifically, we consider a dynamic model economy populated by a large number of ex-ante identical households who can invest in risk-free physical capital and risky human capital. Human capital investment is risky due to idiosyncratic shocks to the stock of household human capital. Households also make an effort choice that is not observable by the government and that affects the probability distribution over idiosyncratic human capital shocks (moral hazard). The government can tax or subsidize capital income and labor income, which affects households' incentives to invest in physical capital and human capital, and it provides social insurance against idiosyncratic human capital (labor income) shocks, which affects households' decisions to apply work effort. The government has to balance its budget period-by-period – there is no government borrowing (no public debt).

The model is made tractable by imposing the following two assumptions. First, human capital investment displays a certain linearity at the household level. Second, preference allow for a time-additive expected-utility representation with a one-period utility function that is additive over consumption and effort. In addition, we assume that the utility function over consumption is logarithmic.[1]

In this paper, we show that these two assumptions yield tractability in the following sense. First, the constrained optimal allocations of the dynamic moral hazard economy can be obtained by solving a static social planner problem – the repeated moral hazard problem has been reduced to a one-shot moral hazard problem. The proof of this tractability results is based on one property of constrained optimal allocations that is of some independent interest: The expected (social) return on human capital investment is equal to the return to physical capital investment for all households with positive human capital investment. Second, the constrained optimal allocations are also equilibrium allocations of a market economy with a simple system of taxes and transfers. Specifically, the government uses transfer payments that are linear in household wealth/income to provide optimal insurance affecting effort incentives and it uses flat-rate income taxes/subsidies to provide optimal investment incentives.

We use our general characterization of optimal social insurance to provide a theoretical and quantitative answer to the questions posed above. Theoretically, we show that a rise in labor market risk will always increase social welfare if the government reacts by adjusting the system of taxes and social insurance optimally.[2] In contrast, if the government does not change the tax and social insurance system, then social welfare may increase or decrease depending on the strength of various adjustment channels. The intuition for this result is straightforward. The increase in labor market risk that individual households have to bear can always be counteracted by the government by increasing social transfer payments so that households are never made worse off. However, an increase in the spread of human capital shocks also means that the labor market provides more opportunities for upward mobility, which can be exploited by the government, through the appropriate adjustment of the tax-

[1] In Section 2.7 we discuss how to extend our approach to general CRRA-utility functions over consumption.

[2] Clearly, this result only holds if a rise in labor market risk is modelled as an increase in the spread of the distribution of human capital shocks keeping the mean fixed. See Section 3.5 for details.

and transfer system, to increase social welfare.

In the quantitative analysis we consider an economy in which job displacement risk is the only source of labor market risk. In this application, human capital investment is best interpreted as on-the-job training and the effort choice corresponds to the decision of an employed worker how hard to work, which affects the probability of being laid off in the case of a mass layoff. We calibrate the process of human capital risk to match the likelihood that a US worker becomes displaced and the long-term earnings losses associated with the displacement event. In line with the analysis in Ljungqvist and Sargent (1998,2008), we simulate a rise in labor market risk by increasing the size of the human capital loss associated with the displacement event. The increase in job displacement risk we feed into the calibrated model economy matches estimates of the empirical literature, which has documented that movements of U.S. workers across occupations and the associated losses of occupation-specific human capital have strongly increased over the 1968-1997 period (Kambourov and Manovskii, 2008 and 2009a).

Our quantitative analysis yields two main results. First, the optimal policy response to the observed rise in job displacement risk is to increase social insurance substantially so that only one fourth of the initial rise in job displacement risk is passed on into consumption. As a consequence, the net effect on work effort and welfare is rather modest – welfare increases by only 0.03 percent of lifetime consumption. Our second result is that the social welfare cost of not adjusting the social insurance system is substantial. Specifically, keeping the generosity of the social insurance system fixed, the observed rise in job displacement risk leads to a substantial increase in the consumption loss of displaced workers and a welfare loss of 0.2 percent of lifetime consumption. In other words, the cost of passive government policy in the face of changing economic conditions is substantial even though individual households can adjust their behavior along three different margins: work effort, physical capital investment (saving), and human capital investment (on-the-job training).

In sum, this paper makes a methodological contribution and an economic contribution. In terms of economic method, we develop a tractable macroeconomic model of moral hazard and show that constrained efficient allocations are simple in the sense that they are characterized by the solution to a static social planner problem. In terms of economic substance, we provide a theoretical and quantitative analysis of the welfare consequences of a rise in labor market risk. We show theoretically that such a rise in labor market risk will always increase social welfare if the government reacts by adjusting the system of taxes and social insurance optimally. Our quantitative application to the rise in job displacement risk in the U.S. shows that the welfare cost of not adjusting policy optimally can be substantial.

Literature. Our paper is related to several strands of the literature. First, there is the literature on the macroeconomic implications of rising labor market uncertainty. Ljungqvist and Sargent (1998, 2008) use an increase in worldwide economic turbulence in the labor market to explain the secular rise in European unemployment starting in the mid 1970s. Heathcote, Storesletten, and Violante (2010) analyze the welfare implications of the secular rise in wage inequality in the US in an incomplete-market model with endogenous skill formation and Krueger and Perri (2004) study the implications for consumption inequality in an economy with endogenously incomplete markets due to limited contract enforcement. Using a model with occupation-specific human capital, Kambourov and Manovskii (2009b) show that the observed rise in occupational mobility in the U.S. can explain a substantial part of the observed rise in wage inequality in the U.S. Finally, Krebs (2003) discusses the growth effects of an increase in labor market risk in an incomplete-market model with endogenous human capital. In contrast to the previous work, in this paper we study the consequences of an increase in labor market risk when moral hazard limits the degree of consumption insurance and economic policy responds optimally to the change in the economic environment.

Second, Rodrik (1998) provides cross-country evidence that more open economies are characterized by a larger government sector and suggests a theoretical interpretation of his finding that is in line with the arguments made here. Specifically, Rodrik (1998) develops a two-period incomplete-market model in which a positive correlation between openness and public sector size arises because more openness leads to more risk households have to bear and the government provides insurance through risk-free pubic services. Interestingly, even though the cross-country evidence provides support for the hypothesis that risk and social insurance are positively correlated, the time series evidence for the U.S. and some other advanced economies over the last 30 years suggests a negative or no correlation. In our concluding remarks we outline an extension of our framework with ex-ante heterogenous households that could potentially explain the observed roll-back of the welfare state that occurred in the U.S. in the 1990s and in Germany in the 2000s.[3]

Third, our work is also related to the macroeconomic literature on optimal taxation in economies with private information. Our theoretical tractability result resembles the results of Farhi and Werning (2007) and Phelan (2006), who show that optimal allocations are the solution to a static social planner problem when the social welfare function puts equal weight on all future generations. In contrast to this work, in the current paper the crucial assumption yielding tractability is not an assumption regarding social preferences, but an assumption about the production structure in combination with an assumption about individual preferences. Our quantitative analysis is also related to previous work on optimal tax policy in private-information economies. See, for example, Farhi and Werning (2012) for an analysis of optimal taxation in economies with physical capital and Kapicka (2015) and Stantcheva (2017) for studies of optimal taxation in models with physical and human capital. The bulk of this literature has considered economies with private information about type (adverse selection) and studied the gains of moving from the actual, inefficient tax system to a new, efficient tax system. In contrast, in this paper we focus on moral hazard and study the optimal response of the tax system to a change in fundamentals moving from one tax system that is efficient before the change in fundamentals to another tax system that is efficient after the change has occurred.[4]

Fourth, our paper relates to the large literature on (constrained) optimal allocations in moral hazard economies. See, for example, Hopenhayn and Nicolini (1997) for a well-known application to optimal unemployment insurance and Laffont and Matrimort (2002)for a survey of micro-oriented literature on moral hazard. Our quantitative analysis is closely related to the work by Pavoni and Violante (2007,2016) on optimal welfare-to-work programs.

[3]For example, in the U.S. "The Personal Responsibility and Work Opportunity Reconciliation Act" was enacted in 1996 and resulted in an overall reduction in the financial assistance for low-income families with children. In Germany, the labor market reforms implemented in 2003-2005, the so-called Hartz reforms, led to a substantial cut in unemployment benefits for long-term unemployed workers.

[4]Specifically, we argue that the available empirical evidence is not sufficient to rule out that the level of (social) insurance against job displacement risk in the U.S. was socially optimal in the 1980s, and we therefore assume that it was optimal. See Section 4 for details and Farhi and Werning (2012) for a similar argument with respect to social insurance against all labor market risk in the U.S.

However, we consider the optimal response of social insurance to a change in fundamentals, whereas Pavoni and Violante (2007,2016) analyze how an inefficient insurance system can be improved for given fundamentals. Our theoretical tractability result echoes the result derived by Holmstrom and Milgrom (1987), Fudenberg, Holmstrom, Milgrom (1990), and Sannikov (2008) for repeated principal-agent problems, but in contrast to these papers we consider a macroeconomic model with an explicit aggregate resources constraint (general equilibrium analysis).

Finally, there is a voluminous literature that studies optimal taxation in incomplete-market models with ad-hoc restrictions on the set of policy instruments. One important issue studied in this literature is to what extent human capital investment should be subsidized. See, for example, Eaton and Rosen (1980) for an early contribution using a two-period model and Krueger and Ludwig (2013) for recent contributions based on a macroeconomic framework. A standard assumption in this literature is that social insurance against human capital risk can only be provided through progressive income taxation that also reduces the expected (after-tax) return to human capital investment. In contrast, the current paper allows the government to use a larger set of policy instruments that are only restricted by the underlying moral hazard friction.

2. Model

This section develops the model, defines the equilibrium concept, and discusses the notion of optimality used in this paper. Specifically, subsections 2.1 and 2.2 describe the fundamentals of the economy, subsections 2.3 to 2.5 define the equilibrium in a market economy, and subsection 2.6 discusses the social planner problem (constrained efficiency). The model combines the incomplete-market model with human capital model developed in Krebs (2003) with a standard model of unobserved effort choice along the lines of Phelan and Townsend (1991) and Rogerson (1985a). The basic framework has ex-ante identical households who face i.i.d. shocks to their human capital and display endogenous growth as in Manuelli and Jones (1990) and Rebelo (1991). In subsection 2.7 we discuss extensions of the model that allow for a general Markov shock process and a more general production structure, and argue that the main tractability result still holds for these extensions.

2.1. Preferences and Uncertainty

The economy has a unit mass of households. Time is discrete and open ended. In each period t, the exogenous part of the individual state of a household is represented by s_t, which captures the effect of idiosyncratic shocks to the human capital of individual households (see below). We denote by $s^t = (s_1, \ldots, s_t)$ the history of exogenous shocks up to period t. We assume that the probability of history $s^t = (s_1, \ldots, s_t)$ occurring is given by $\pi_t(s^t|e^{t-1}) = \pi(s_t|e_{t-1}) \times \ldots \pi(s_1|e_0)$, where e_n is the effort taken by the household in period n and $\pi(s_n|e_{n-1})$ is the probability of state s_n given effort choice e_{n-1}. In other words, for given effort plan, $\{e_t\}$, the random variables s_t and s_{t+n} are independently distributed for all t and n.

To streamline the exposition, we assume that there are a finite number of realizations, $s_t \in \{1, \ldots, S\}$. We further assume that $e_t \in \mathbb{R}_+$ and that $\pi(s, .)$ is continuously differentiable, strictly increasing, and concave for each s. Finally, we assume that π satisfies the monotone-likelihood ratio condition (Rogerson, 1985b).

Households are risk-averse and have identical preferences that allow for a time-additive expected utility representation with one-period utility function that is additive over consumption and effort and logarithmic over consumption. Let $\{c_t, e_t|s_0\}$ stand for the consumption-effort plan of a household of initial type s_0. Expected lifetime utility associated with the consumption-effort plan $\{c_t, e_t|s_0\}$ is then given by

$$U(\{c_t, e_t|s_0\}) = \sum_{t=0}^{\infty} \sum_{s^t|s_0} \beta^t \left[\ln c_t(s^t) - d(e_t(s^t))\right] \pi_t(s^t|e^{t-1}(s^{t-1})) \tag{1}$$

where β is the pure discount factor and $d(.)$ is a dis-utility function that is continuously differentiable, strictly increasing and strictly convex.

2.2. Production

There is one consumption good that is produced using the aggregate production function

$$Y_t = F\left(K_t, H_t\right) \ , \tag{2}$$

where Y_t is aggregate output in period t, K_t is the aggregate stock of physical capital employed in production, and H_t is the aggregate stock of human capital employed in production.

We assume that F is a standard neoclassical production function. In particular, F displays constant returns to scale with respect to the two input factors physical capital, K, and human capital, H.

The consumption good can be transformed into the physical capital good one-for-one. In other words, production of the consumption good and production of physical capital employ the same production function, F. The consumption good is perishable and physical capital depreciates at a constant rate, δ_k.

Human capital is produced at the household level. An individual household can transform the consumption good into human capital using a quantity of x_{ht} consumption goods to produce ϕx_{ht} units of human capital. Note that $1/\phi$ is the price of human capital in units of the consumption (physical capital) good. Existing human capital is subject to random shocks, $\eta_t = \eta(s_t)$. The production function and law of motion for household-level human capital, h_t, are described by

$$\begin{aligned} h_{t+1} &= (1+\eta(s_t))h_t + \phi x_{ht} \\ x_{ht} &\geq 0 \end{aligned} \tag{3}$$

Note that h_{t+1} is a linear function of x_{ht} and that we impose a non-negativity constraint on human capital investment. Note further that equation (3) holds for all t and s^t, but for notational ease we suppress the dependence on s^t.

The η-term in the human capital accumulation equation (3) represents changes in human capital that are affected by effort choices and do not require (substantial) goods investment. For example, positive human capital growth, $\eta(s) > 0$, can represent learning-by-doing, and in this case $\pi(., e)$ summarizes the effect of work effort on the success of on-the-job learning. Job-to-job transition is a second example of a positive human capital shock, and in this case it is (on-the-job) search effort that determines the likelihood that the positive realization occurs (the search is successful). In contrast, job loss and the associated loss of (firm or sector specific human capital) is a typical example of a negative realization $\eta(s) < 0$. In this case, $\pi(., e)$ may represent both the effect of work effort on the likelihood of job loss and the effect of search effort during unemployment on the size of human capital loss associated with the job loss. In our quantitative analysis conducted in sections 4 and 5, we focus on job displacement risk as the only source of human capital risk and interpret the negative shock to human capital as the loss of firm- or occupation-specific human capital associated with the displacement event.[5]

The term x_{ht} in equation (3) represents changes to human capital that require goods investment. Formal education is a typical example, in which case construction of school buildings, the use of teaching material, and the salaries of teachers all are part of the goods cost of human capital production. Equation (3) neglects the use of time in human capital production. In section 2.7 below we discuss extensions of the model in which human capital production also requires time as an input, which happens when parents spend time with their school children or adults decide how much of their time to spend for in formal education (college, professional school) and how much time to spend working.

2.3. Household decision problem

We next describe the decision problem of households in a market economy. We consider sequential equilibria. Specifically, at time $t = 0$, an individual household begins life in initial state s_0 and with initial endowment (a_0, h_0), where a_0 is the amount of financial asset holding of the household in period $t = 0$. To ease the notation, we assume that the initial asset holding of an individual household are proportional to the initial human capital of the household: $a_0 = \frac{K_0}{H_0} h_0$. Thus, the initial state/type of an individual household is given by (h_0, s_0). The initial state of the economy is defined by an initial distribution of individual households over types, $\pi_0(h_0, s_0)$, and an initial aggregate stock of physical capital, K_0. Note that taking the expectations over h_0, respectively a_0, using π_0 yields the initial aggregate stock of human capital, H_0, respectively physical capital, K_0.

A household of initial type (h_0, s_0) chooses a plan consisting of a sequence of functions $\{c_t, e_t, a_{t+1}, h_{t+1} | h_0, s_0\}$, where each $(c_t, e_t, a_{t+1}, h_{t+1})$ stands for a function mapping individual histories s^t into a choice of consumption, $c_t(s^t)$, effort, $e_t(s^t)$, financial asset holding, $a_{t+1}(s^t)$, and human capital, $h_{t+1}(s^t)$. Note that the choice of an action $(c_t, e_t, a_{t+1}, h_{t+1})$ amount to an effort decision, a consumption-saving decision, and a decision how to allocate

[5] We use $\eta(s_t)$ instead of $\eta(s_{t+1})$ in (3) in order to simplify the formal proofs, a timing choice also made in Krebs (2003) and Stantcheva (2017). However, the current analysis and results apply, mutatis mutandis, if the timing is changed and $\eta(s_{t+1})$ is used in (3). See Stokey and Lucas (1989) for a general discussion of this issue in choice problems under uncertainty.

the saving between investment in financial assets and investment in human capital.

An individual household with financial asset holding a_t in period t receives financial income $r_f a_t$, where r_f is the risk-free real interest rate (the return to financial investments). A household with human capital h_t earns labor income $r_h h_t$, where r_h is the wage rate (rental rate) per unit of human capital. Note that investment of one unit of the consumption good in financial capital yields the risk-free return r_f and investment of one unit of the consumption good in human capital earns the risky return $\phi r_h + \eta(s_t)$. Note further that we confine attention to wage rates and interest rates that are independent of time.

The government chooses a system of taxes and transfers that provides insurance and incentives. This tax-and-transfer system consists of a capital income tax/subsidy, $\tau_a r_f a_t$, a labor income (human capital) tax/subsidy, $\tau_h r_h h_t$, and transfer payments $tr(s_t) r_{ht} h_t$. Note that taxes/subsidies and transfer payments are linear in the choice variables k and h. Further, we assume that capital and labor income taxes/subsidies are constant over time and independent of individual histories and that transfer payments only depend on the current shock realization: $tr_t = tr(s_t)$. A *tax-and-transfer policy* is a triple (τ_k, τ_h, tr), where τ_k and τ_h are real numbers and tr is a function, $tr_t = tr(s_t)$.

The household budget constraint reads:

$$\begin{aligned} c_t + a_{t+1} - a_t + x_{ht} &= (1 - \tau_h + tr(s_t)) r_h h_t + (1 - \tau_a) r_f a_t \qquad (4) \\ x_{ht} \geq 0 \ ; \ \ a_{t+1} + \frac{h_{t+1}}{\phi} \geq 0 & \end{aligned}$$

The budget constraint (4) has to hold for all t and s^t, but for notational ease we have suppressed the dependence on s^t. Note the human capital equation (3) in conjunction with the non-negativity constraint on human capital investment, $x_{ht} \geq 0$, implies that human capital is always strictly positive: $h_{t+1} > 0$. Note also that the budget constraint (4) is linear in the household choice variables a and h.

For given tax-and-transfer policy, (τ_k, τ_h, tr), and given rental rates, r_f and r_h, an individual household of initial type, (s_0, h_0), chooses a plan, $\{c_t, e_t, a_{t+1}, h_{t+1} | h_0, s_0\}$, that solves the utility maximization problem:

$$\max_{\{c_t, e_t, a_t, h_t | h_0, s_0\}} U(\{c_t, e_t | s_0\}) \qquad (5)$$

$$subject\ to\ :\ \{c_t, e_t, a_{t+1}, h_{t+1}|h_0, s_0\} \in B(h_0, s_0)$$

where the budget set, $B(h_0, s_0)$, of an household of type (h_0, s_0) is defined by equations (3) and (4) and the expected lifetime utility, U, associated with a consumption-effort plan, $\{c_t, e_t|s_0\}$, is defined in (1).

2.4. Firm decision problem

The consumption good is produced by a representative firm that rents physical capital, K_t, and human capital, H_t, in competitive markets at rentals rates r_k and r_h, respectively. In each period t, the representative firm rents physical and human capital up to the point where current profit is maximized:

$$\max_{K_t, H_t} \{F(K_t, H_t) - r_k K_t - r_h H_t\} \tag{6}$$

2.5. Market Equilibrium

We now define a sequential market equilibrium. There is a financial sector that can transform household saving into physical capital at no cost. Thus, the no-arbitrage condition

$$r_f \;=\; r_k - \delta_k \tag{7}$$

has to hold and household financial capital, $E[a_t]$, is also the physical capital supplied to firms, K_t. We consider a closed economy so that in equilibrium the demand for capital and labor by the representative firm must be equal to the corresponding aggregate supply by all (domestic) households:

$$\begin{aligned} K_t &= E[a_t] \\ H_t &= E[h_t] \end{aligned} \tag{8}$$

Note that we assume that an appropriate law of large numbers applies so that aggregate household variables are obtained by taking the expectations over all individual histories and initial types: $E[a_t] = \sum_{h_0, s_0, s^{t-1}} a_t(h_0, s_0, s^{t-1})\pi_t(s^{t-1}, e^{t-1}(s^{t-1})|h_0, s_0)\pi_0(h_0, s_0)$ and $E[h_t] = \sum_{h_0, s_0, s^t} h_t(h_0, s_0, s^{t-1})\pi_t(s^t, e^{t-1}(h_0, s_0, s^{t-1})|h_0, s_0)\pi_0(h_0, s_0)$.

We assume that the government runs a balanced budget in each period. We further assume that the social insurance system has its own budget that balances in each period:

$$\tau_a r_f E[a_t] \,+\, \tau_h r_h E[h_t] \;=\; 0 \tag{9}$$

$$E[tr(s_t)] = 0$$

Note that in the current setting the two government budget constraints (9) are equivalent to one consolidated budget constraint in the sense that the same set of equilibrium allocations can be achieved (see proposition 3 below). However, we prefer to work with the two government budget constraints (9) to separate the tax system, which changes investment incentives, from the social insurance system, which changes the incentive to apply effort.

Recall that an individual household of initial type s_0 choose a household plan $\{c_t, e_t, a_{t+1}, h_{t+1}|h_0, s_0\}$. We denote the family of household plans, one for each household type (h_0, s_0), by $\{c_t, e_t, a_{t+1}, h_{t+1}\}$. Note that a family of household plans also defines an allocation. Our definition of a market equilibrium is standard:

Definition 1. A *sequential market equilibrium* for given tax-and-transfer policy, (τ_k, τ_h, tr), is a family of household plans, $\{c_t, e_t, a_{t+1}, h_{t+1}\}$, a plan for the representative firm, $\{K_t, H_t\}$, an interest rate, r_f, and a wage rate, r_h, so that i) for each household type (h_0, s_0) the plan $\{c_t, e_t, k_t, h_t|h_0, s_0\}$ solves the household's utility maximization problem (5), ii) $\{K_t, H_t\}$ solves the firm's profit maximization problem (6) in each period t, iii) the market clearing conditions (8) and the no-arbitrage condition (7) hold, and iv) the government budget constraint (9) is satisfied.

Aggregate physical capital and aggregate human capital evolve according to

$$\begin{aligned} K_{t+1} &= (1-\delta_k)K_t + X_{kt} \\ H_{t+1} &= H_t + E[\eta_t h_t] + \phi X_{ht} \end{aligned} \tag{10}$$

where $X_{kt} = E[a_{t+1}] - E[a_t]$ is aggregate investment in physical capital (aggregate saving) and X_{ht} is aggregate goods investment in human capital. Note that $E[\eta_t h_t] \neq E[\eta_t]E[h_t]$ as long as e_{t-1} depends on s^{t-1}. However, below we see that e_{t-1} is independent of s^{t-1} in equilibrium and for optimal allocations, in which case the term $E[\eta_t h_t]$ can be replaced by $E[\eta_t]H_t$ in equation (10).

The factor market clearing conditions (8) and the no-arbitrage-condition (7) together with the government budget constraint (9) and the individual budget constraint (4) imply the following aggregate resource constraint (Walras' law):

$$C_t + X_{kt} + X_{ht} = Y_t \quad . \tag{11}$$

In other words, goods market clearing has to hold: Aggregate output produced is equal to the sum of aggregate consumption, aggregate investment in physical capital, and aggregate goods investment in human capital.

We see below (proposition 1) that in a sequential market equilibrium aggregate ratio variables, such as the aggregate capital-to-labor ratio and the aggregate capital-to-output ratio, are constant over time, but aggregate level variables, such as aggregate output, grow without bounds over time. The property of unbounded equilibrium growth (endogenous growth) is an implication of the constant-returns-to-scale assumption in combination with the assumption that the two input factors, physical capital and human capital, can be accumulated without limits. In subsection 2.7 we discuss two extensions of the model that make equilibrium output bounded.

2.6. Optimal Allocations

To define (constrained) optimal allocations, we consider a social planner who directly chooses an allocation, $\{c_t, e_t, h_{t+1}, K_{t+1}\}$ with $H_{t+1} = E[h_{t+1}]$, subject to an aggregate resource constraint defined by (2), (10), and (11) and an incentive compatibility constraint that arises because effort choices are private information. Specifically, the social planner can only choose consumption-effort plans, $\{c_t, e_t|h_0, s_0\}$, that are *incentive compatible* in the sense that households will adhere to the proposed effort plan, that is, $\{c_t, e_t|h_0, s_0\}$ has to satisfy:

$$\forall\,(h_0, s_0, s^t)\,,\ \forall\,\{\hat{e}_{t+n}|h_0, s_0, s^t\}\ : \tag{12}$$

$$U_t(\{c_{t+n}, e_{t+n}|h_0, s_0, s^t\})\ \geq\ U_t(\{c_{t+n}, \hat{e}_{t+n}|h_0, s_0, s^t\})\ .$$

where $\{c_{t+n}, e_{t+n}|h_0, s_0, s^t\}$ denote the continuation plan at (h_0, s_0, s^t) and U_t the corresponding continuation utility. We define the constraint set of the social planner problem as

$$\mathbf{A} \equiv \{\{c_t, e_t, h_{t+1}, K_{t+1}\}|\{c_t, e_t, h_{t+1}, K_{t+1}\}\ satisfies\ (2), (10), (11),\ and\ (12)\}\ . \tag{13}$$

We assume that the social planner's objective function is social welfare defined as the weighted average of the expected lifetime utility of individual households defined in (1), where we use the Pareto weight μ_0, to weigh the importance of households of type (h_0, s_0).

For notational simplicity, we assume a finite number of initial types. If $\mu(h_0, s_0) = \pi_0(h_0, s_0)$, then each individual household is assigned equal importance by the social planner.

Definition 2. An *optimal allocation* is the solution to the social planner problem

$$\max_{\{c_t, e_t, h_{t+1}, K_{t+1}\}} \sum_{h_0, s_0} U(\{c_t, e_t | h_0, s_0\})\mu(h_0, s_0) \tag{14}$$

$$\textit{subject to} \; : \; \{c_t, e_t, h_{t+1}, K_{t+1}\} \; \in \; \mathbf{A}$$

where the constraint set $\mathbf{A}$ is defined in equation (13).

Note that in our discussion of optimal allocations we only use the aggregate physical capital stock, K, in the definition of an allocation. The distribution of physical capital across households is irrelevant since only the aggregate level of physical capital enters into the production equations. In contrast, human capital is produced at the household level and the allocation of human capital across households is therefore specified as part of an allocation. There is, however, also a considerable degree of indeterminacy with respect to the optimal allocation of individual human capital because of the linearity of the individual accumulation (production) equation for human capital, which we discuss in more detail below.

Our definition of an optimal allocation assumes that the social planner can observe individual human capital h. Similarly, our definition of sequential market equilibria assumes that the government can observe capital and labor income and levi a tax (pay a subsidy) on theses two sources of income. In adverse selection economies in which there is private information about type realizations, s^t, this assumption would give rise to a certain inconsistency in the sense that the realization of s_t can be inferred from the observation of h see Mirrlees (1971) for a classical discussion of this point. However, in the moral hazard economy considered in this paper, there is no inconsistency since effort, e, affects only probabilities and information about the particular value of h (the realization of s) does not allow one to infer the value of e. Note that the assumptions about observability made in this paper are standard in the moral hazard literature (Laffont and Martimort, 2002) and coincide with assumptions made in Da Costa and Maestri (2007) and Stantcheva (2017) in their studies of adverse selection economies with human capital investment. In contrast, Abraham and Pavoni (2008) consider a moral hazard economy with hidden financial wealth and Kapicka

(2015) studies an adverse selection economy with unobservable human capital investment.

2.7. Extensions

There are several extensions of the basic framework that can be incorporated without sacrificing the tractability of the model. Specifically, the main characterization results (propositions 1-4) still hold, mutatis mutandis, and proofs of the various characterization result are similar to the ones given in this paper. In this subsection, we briefly discuss some of these extensions.

First, the assumption of i.i.d. human capital shocks can replaced by the assumption that $\{s_t\}$ follows a general Markov process. Clearly, in this case effort and portfolio choices will depend on the current shock realization, but not on past realizations of shocks or on initial states. In addition, the shock, s_t, might affect the productivity of human capital production, the efficiency of existing human capital in producing output, the utility of consumption, or the dis-utility of effort. See Krebs, Kuhn, and Wright (2015) for a limited-enforcement version of the model with a large degree of household heterogeneity due to a rich shock structure.

Second, as in Krebs (2003) and Stantcheva (2017), equation (3) assumes that human capital production only uses goods. In contrast, Guvenen et al. (2014), Heckman et al. (1998), and Huggett et al. (2011) focus on the time investment in human capital. Clearly,in most cases human capital investment uses both goods and time. The tractability result derived in this paper also holds for the case in which both goods and time are used to produce human capital as long as there is constant-returns-to-scale. Specifically, we can introduce a time cost of human capital production by replacing the term ϕx_{ht} in (3) by $\phi\left(h_t l_t\right)^\rho x_{xt}^{1-\rho}$, where l_t denotes the time spend in human capital production. If there is a fixed amount of time that is allocated between producing human capital, l_t, and working, $1-l_t$, it is straightforward to show that this human capital production function gives rise to a human capital accumulation equation (3) that is still linear in x_{ht} after substituting out the optimal choice of l_t. Though the main results of this paper also hold for this case, the decentralization of optimal allocations (proposition 4) requires one additional tax instrument since there is one additional choice variable.

Third, the non-negativity constraint on human capital investment can be relaxed. Specifically, our theoretical results also hold if we replace $x_{ht} \geq 0$ by h_{t+1}, in which case the condition (21) is not required to hold. However, this generalization comes at a cost in terms of economic interpretation, namely that the model allows for equilibrium/optimal allocations with negative human capital investment (human capital is "sold" in certain states).

Fourth, as in Jones and Manuelli (1990) and Rebelo (1991), the aggregate production function (2) displays constant-returns-to-scale with respect production factors that can be accumulated without bounds, a property that is well-known to generate endogenous growth. The main results of this paper still hold if (2) is replaced by a production function with diminishing returns or, equivalently, a production function with constant-returns-to-scale and a third, fixed factor of production (land). However, in this case we have an explicit time-dependence of individual and aggregate variables, and convergence towards a steady state instead of unbounded growth under certain conditions.

Fifth, the assumption of infinitely-lived households (dynasties) can be replaced by an overlapping-generations structure in which households die stochastically and in each period new-born households are injected into the economy. If new-born households begin life with an endowment of human capital that is proportional to aggregate human capital, as in Krebs, Kuhn, and Scheffel (2015), then the endogenous-growth nature of the model is preserved. In contrast, if the distribution of human capital of new-born households has a fixed mean that is independent of the existing stock of human capital, then aggregate output remains bounded even with the production function (2) and under certain conditions there is convergence towards a steady state.

Finally, there is the question how the current analysis can be generalized to preferences that are not necessarily logarithmic over consumption. For the analysis of equilibria of an incomplete-market economy, it is straightforward to show that a version of proposition 1 still holds if the one-period utility function is given by $\frac{c^{1-\gamma}}{1-\gamma}\nu(e)$ in the sense that consumption is still a linear function of total wealth and portfolio choices are identical across households, where ν is a function decreasing in e. However, the proof of the result that optimal allocations are simple requires additive one-period utility functions, $u(c, e) = u(c) - d(e)$. This rules out balanced growth for any utility function except the logarithmic function, but is an assumption common in the literature on optimal taxation with private information (Golosov et al., 2003). The extension of the optimality analysis to utility functions beyond the logarithmic function is an important topic for future research.

3. Theoretical Results

This section states and discusses the theoretical results. Subsection 3.1 provides a full characterization of equilibria of the market economy (proposition 1). The next subsection gives a first characterization of optimal allocations: Expected social returns on human capital investment have to be equal the risk-free rate for all households with positive levels of human capital investment (proposition 2). Subsection 3.3 provides a full characterization of optimal allocations: The dynamic social planner problem of the infinite-horizon economy can be reduced to a static social problem of a one-period economy (proposition 3). Subsection 3.4 characterizes the tax-and-transfer systems that yield market equilibria with optimal allocations (proposition 4). The final subsection shows that an increase in human capital risk always increases social welfare if the tax-and-transfer system is optimally adjusted (proposition 5). Proofs of the propositions are collected in the Appendix.

3.1. Equilibrium Allocations

We begin with a convenient characterization of the solution to the firm's problem. Under constant-returns-to-scale, profit maximization (6) implies that

$$\begin{aligned} r_{kt} &= F_k(\tilde{K}_t) \\ r_{ht} &= F_h(\tilde{K}_t) \end{aligned} \tag{15}$$

where $\tilde{K}_t = \frac{K_t}{H_t}$ is the ratio of aggregate physical capital to aggregate human capital (capital-to-labor ratio) and $F_k(\tilde{K}_t)$ and $F_h(\tilde{K}_t)$ stand for the marginal product of physical capital and human capital, respectively. Equation (15) summarizes the implications of profit maximization by the representative firm.

We next turn to the household problem. To this end, it is convenient to introduce the following new household-level variables:

$$w_t = k_t + \frac{h_t}{\phi} \ , \ \theta_t = \frac{k_t}{w_t} \ , \ 1 - \theta_t = \frac{h_t}{\phi w_t} \tag{16}$$

$$r_t \;=\; \theta_t(1-\tau_k)\left(F_k(\tilde{K}_t)-\delta_k\right) + (1-\theta_t)\left((1-\tau_h+tr(s_t))\phi\, F_h(\tilde{K}_t)+\eta(s_t)\right)$$

Here w_t is the value of total wealth, financial and human, measured in units of the consumption good, θ_t the share of total wealth invested in financial capital (financial asset holding), and $(1-\theta_t)$ is the share of total wealth invested in human capital. The expression $1+r$ is the total return on investing one unit of the consumption good. Note further that w_t is total wealth before asset have paid off and depreciation has taken place and $(1+r_t)w_t$ is total wealth after asset payoff and depreciation has occurred.

Using the change-of-variables (16), we can rewrite the budget constraint (4) as:

$$\begin{aligned} w_{t+1} \;&=\; (1+r_t(\theta_t,\tilde{K}_t,s_t))w_t - c_t \\ & w_{t+1}\geq 0 \;\;;\;\; (1-\theta_{t+1})w_{t+1}\geq (1+\eta(s_t))(1-\theta_t)w_t \end{aligned} \tag{17}$$

Note that the second inequality constraint in (17) is the non-negativity constraint on human capital investment. Clearly, (17) is the budget constraint associated with a consumption-saving problem and a portfolio choice problem when there are two investment opportunities, namely risk-free financial capital and risky human capital. The risk-free return to financial capital investment is given by $(1-\tau_k)\left(F_k(\tilde{K}_t)-\delta_k\right)$ and the risky return to human capital investment is $(1-\tau_h+tr(s_t))\phi F_h(\tilde{K}_t)+\eta(s_t)$. Note that the total investment return, r_t, depends on the individual portfolio share, θ_t, the aggregate capital-to-labor ratio $\tilde{K}_t$, which captures any general equilibrium effects, and the individual shock s_t, which represents human capital risk. The investment return also depends on the tax-and-transfer rates, $(\tau_k,\tau_h,tr(.))$, but for notational ease this dependence is suppressed in (17).

A household plan is now given by $\{c_t,e_t,w_{t+1},\theta_{t+1}|w_0,s_0\}$, where $(c_t,e_t,w_{t+1},\theta_{t+1})$ is a function that maps histories of shocks, s^t, into choices $(c_t(s^t),e_t(s^t),w_{t+1}(s^t),\theta_{t+1}(s^t))$. The definition of a sequential equilibrium using household plans $\{c_t,e_t,w_{t+1},\theta_{t+1}|w_0,s_0\}$ instead of $\{c_t,e_t,a_{t+1},h_{t+1}|h_0,s_0\}$ is, mutatis mutandis, the same as definition 1.

The household decision problem has a simple solution. Specifically, current consumption, c_t, and next period's wealth, w_{t+1}, are linear functions of current wealth, w_t, given by

$$\begin{aligned} c_t(s^t) \;&=\; (1-\beta)(1+r(\theta,\tilde{K},s_t))w_t(s^{t-1}) \\ w_{t+1}(s^t) \;&=\; \beta(1+r(\theta,\tilde{K},s_t))w_t(s^{t-1})\;, \end{aligned} \tag{18}$$

where portfolio and effort choice are the solution to the static household maximization problem:

$$\max_{\theta,e}\left\{-d(e)\ +\ \frac{\beta}{1-\beta}\sum_s \ln(1+r(\theta,\tilde{K},s))\pi(s,e)\right\} \tag{19}$$

Note that in (19) we assume that the aggregate capital-to-labor ratio, $\tilde{K}$, is constant over time – a conjecture that turns out to be correct in equilibrium. Clearly, equation (19) implies that all households make identical portfolio and effort choices.

The linearity of individual consumption and individual wealth choices means that aggregate market clearing reduces to the condition that the (common) portfolio choice of households, θ, has to be consistent with the capital-to-labor ratio chosen by the firm, $\tilde{K}$. More precisely, let $\theta = \theta(\tilde{K})$ be the portfolio demand function defined by the solution to (19) for varying $\tilde{K}$. The two market clearing conditions (8) hold if

$$\tilde{K} \ = \ \frac{\theta(\tilde{K})}{\phi(1-\theta(\tilde{K}))} \tag{20}$$

Equation (20) is derived from (8) using $k = \theta w$ and $h = \phi(1-\theta)w$ and the fact that because of the constant-returns-to-scale assumption the two equations in (8) can be reduced to one equation.

The static household maximization problem (19) does not impose the non-negativity constraint on human capital investment in (17). This non-negativity constraint holds in equilibrium if

$$\beta\left(1+r(\theta(\tilde{K}),\tilde{K},s)\right)\ \geq\ 1+\eta(s) \tag{21}$$

for all s.

In summary, we have the following characterization of equilibria of the market economy:

Proposition 1. Let $\tilde{K}^*$ be the solution to the equation (20), where the portfolio function $\theta = \theta(\tilde{K})$ is the solution to the static household maximization problem (19). Let $\theta^* = \theta(\tilde{K}^*)$ and e^* be the corresponding portfolio choice and effort choice and assume that condition (21) holds at $(\tilde{K}^*, \theta^*)$. Then the triple $(\tilde{K}^*, \theta^*, e^*)$ defines a simple sequential market equilibrium. More precisely, in equilibrium the aggregate capital-to-labor ratio is constant over time, $\tilde{K}_t = \tilde{K}^*$, and household portfolio and effort choices are time- and history-independent,

$\theta_{t+1}(s^t) = \theta^*$, and $e_t(s^t) = e^*$. Further, individual consumption and individual wealth evolve according to (18) and expected lifetime utility of households is given by:

$$
\begin{aligned}
U\left(\{c_t, e_t | w_0, \theta_0, s_0\}\right) &= \frac{1}{1-\beta}\left[\ln(1-\beta) + \frac{\beta}{1-\beta}\ln\beta + \ln(1 + r(\theta_0, \tilde{K}_0, s_0)) + \ln w_0 - d(e^*)\right] \\
&\quad + \frac{\beta}{(1-\beta)^2}\sum_s \ln(1 + r(\theta^*, \tilde{K}^*, s))\pi(s, e^*) \; .
\end{aligned}
$$

Proposition 1 is the generalization of the tractability result of Krebs (2003) to incomplete-market models with an effort choice. The representation of equilibrium welfare in proposition 1 uses (w_0, θ_0, s_0) as a description of the initial state of an individual household. Using the definition $w_0 = a_0 + h_0/\phi$ and $\theta_0 = \frac{a_0}{a_0 + h_0/\phi}$ and the assumption $a_0 = \frac{K_0}{H_0}h_0$, we can use proposition 1 to find the corresponding formula for $U\left(\{c_t, e_t | h_0, s_0\}\right)$.

Given our assumptions, we can use the first-order condition approach to find the solution to the static utility maximization problem (19). These first-order conditions read:

$$
\begin{aligned}
0 &= \sum_s \frac{(1 - \tau_h + tr(s))\phi r_h(\tilde{K}) + \eta(s) - (1-\tau_f)r_f(\tilde{K})}{1 + r(\theta, \tilde{K}, s)}\pi(s, e) \qquad (22) \\
d'(e) &= \frac{\beta}{1-\beta}\sum_s ln\left(1 + r(\theta, \tilde{K}, s)\right)\frac{\partial \pi}{\partial e}(s, e)
\end{aligned}
$$

The first equation in (22) expresses the optimal portfolio choice of individual households. It states that the expected marginal utility weighted excess return of human capital investment over physical capital investment must be zero, where the marginal utility is represented by the term $(1 + r)^{-1}$. The second equation in (22) is the first-order condition with respect to the effort choice and says hat the dis-utility of increasing effort is equal to the expected gains associated with an increase in effort.

To gain a better understanding of the way the social insurance system, $tr(.)$, affects individual consumption and therefore welfare, consider the evolution of individual consumption that follows from proposition 1:

$$
c_{t+1}(s^{+1}) = \beta\left(1 + \theta(1-\tau_k)r_f + (1-\theta)\left((1 - \tau_h + tr(s_{t+1}))\phi F_h + \eta(s_{t+1})\right)\right)c_t(s^t) \qquad (23)
$$

Individual consumption grows at a rate that is equal to $\beta(1 + r)$, where the total investment returns, r, depends on portfolio choice, θ, financial returns, $r_f = F_k - \delta_k$, human capital returns $\phi\, F_h$, ex-post shocks, $\eta(s_t)$, the tax rates, τ_a and τ_h, and the transfer payments (insurance), $tr(s_t)$. From (23) we immediately conclude that consumption is independent of human capital shocks if $tr(s_{t+1})\phi\, F_h = -\eta(s_{t+1})$. This is intuitive since in the case of a negative human capital shock, $\eta(s_t)-\bar{\eta}(e) < 0$, the term $(1-\theta)\eta(s_t)w_t < 0$ is the total amount of human capital lost in units of the consumption good and the term $(1-\theta)tr(s_{t+1})\phi\, r_h w_t > 0$ is the corresponding transfer payment in consumption units, where we used the notation $\bar{\eta}(e) = \sum_s \eta(s)\pi(s,e)$.

Proposition 1 characterizes equilibria for given tax-and-transfer policy. The government budget constraint (9) is satisfied if (and only if) the condition

$$\tau_k \tilde{K}\left(F_k(\tilde{K}) - \delta_k\right) \;+\; \tau_h F_h(\tilde{K}) \;=\; 0 \tag{24}$$

holds. Clearly, equation (24) imposes a further condition that determines the set of budget-feasible government policies (τ_k, τ_h, tr). Note that an equilibrium allocation defined in proposition 1 only satisfies the aggregate resource constraint (9) if the government budget constraint (24) is satisfied.

Proposition 1 in conjunction with the balanced-budget condition (24) provide a convenient equilibrium characterization that has two useful properties. First, the consumption-saving choice is linear in wealth and the portfolio and effort choice are constant and independent of wealth (histories). Second, the equilibrium can be computed without the knowledge of the endogenous, infinite-dimensional wealth distribution. These two properties render the computation of equilibria extremely simple since it suffices to solve the equation system defined by (20), (22), and (24) – four equations in four unknowns, namely $(e, \theta, \tilde{K})$ plus one tax parameter.

Proposition 1 shows how the household-level variables evolve in equilibrium. The evolution of aggregate variables is obtained by taking the expectations over individual variables using the government budget constraint (24):

$$\begin{aligned}
C_t &= (1-\beta)\left[1 + \frac{\phi\tilde{K}^*}{1+\phi\tilde{K}^*}\left(r_k(\tilde{K}^*) - \delta_k\right) + \frac{1}{1+\phi\tilde{K}^*}\left(\phi r_h(\tilde{K}^*) + \bar{\eta}(e^*)\right)\right] W_t \quad (25)\\
W_{t+1} &= \beta\left[1 + \frac{\phi\tilde{K}^*}{1+\phi\tilde{K}^*}\left(r_k(\tilde{K}^*) - \delta_k\right) + \frac{1}{1+\phi\tilde{K}^*}\left(\phi r_h(\tilde{K}^*) + \bar{\eta}(e^*)\right)\right] W_t
\end{aligned}$$

$$K_t \;=\; \frac{\phi\tilde{K}^*}{1+\phi\tilde{K}^*}W_t \;\;;\;\; H_t \;=\; \frac{1}{1+\phi\tilde{K}^*}\,W_t\;,$$

where we used the notation $\bar{\eta}(e^*) \doteq \sum_s \eta(s)\pi(s, e^*)$.

3.2. Optimal Allocations: Production Efficiency

Consider an allocation $\{c_t, e_t, h_{t+1}, K_{t+1}\}$. In economies with complete information, production efficiency requires that (expected) social returns on alternative investment opportunities are equalized if investment levels are positive.[6] In the model considered in this paper, this equalization-of-returns condition reads:

$$\phi\, F_h(\tilde{K}_{t+1}) \;+\; \sum_{s_{t+1}} \eta(s_{t+1})\pi(s_{t+1}|e_t(h_0, s_0, s^t)) \;=\; F_k(\tilde{K}_{t+1}) - \delta_k \tag{26}$$

Proposition 2 below shows that the optimality condition (26) also characterize optimal allocations of moral hazard economies for all initial states, (h_0, s_0), and all histories, s^t, with positive human capital investment, $x_{ht}(h_0, s_0, s^t) > 0$. Clearly, the efficiency condition (26) does not have to hold for histories with $x_{ht}(h_0, s_0, s^t) = 0$. However, even for those histories an inequality version of (26) holds: Expected human capital returns cannot exceed the return to physical capital investment. In addition, a standard argument shows that the optimal $\tilde{K}_t$ is independent of t since production displays constant returns to scale with respect to H and K, and these two factors of production can be adjusted at no cost. Thus, we have the following result:

Proposition 2. An optimal allocation exists. The optimal aggregate capital-to-labor ratio is constant over time: $\tilde{K}_t = \tilde{K}$ for all periods $t = 1, \ldots$. Further, for all initial states, (h_0, s_0), and all histories, s^t, the expected return on human capital investment cannot exceed the return on physical capital investment:

$$\phi\, F_h(\tilde{K}) \;+\; \sum_{s_{t+1}} \eta(s_{t+1})\pi(s_{t+1}|e_t(h_0, s_0, s^t)) \;\leq\; F_k(\tilde{K}) - \delta_k\;, \tag{27}$$

where (27) holds with equality for all (h_0, s_0, s^t), with positive human capital investment, $x_{ht}(h_0, s_0, s^t) > 0$.

---

[6] More precisely, if a capital allocation maximizes aggregate output net of depreciation, then the (expected) returns on physical capital investment and human capital investment are equalized. Further, the capital-to-labor ratio that maximizes the expected total investment return for given effort level is determined by the equality-of-returns condition.

The proof of proposition 2 is quite general and does not depend on the linearity of individual human capital investment opportunities. Indeed, the proof conducted in the Appendix shows that the result holds for any production function (2) and any human capital accumulation equation of the type $h_{t+1} = g(h_t, x_{ht}, l_t, s_t)$ as long as financial investment (borrowing and lending) and human capital investment (labor income) are observable, where l_t is the time spent in human capital production. For the general case the human capital return has to be defined as $r_{h,t+1} = g_{x_{ht}} \left((1 - l_{t+1}) F_{h,t+1} + g_{h,t+1}/g_{x_{h,t+1}} \right) - 1$. The result that an efficiency condition of the type (26) has to hold even in private information economies as long as investment choice are observable has first been shown in Da Costa and Maestri (2007) in a one-period model with private information about types.

One direct implication of proposition 2 is that effort choices are the same for all initial states and all histories with positive human capital investment: $e_t(h_0, s_0, s^t) = e^*$ for all (h_0, s_0, s^t) with $x_{ht}(h_0, s_0, s^t) > 0$. This follows since different effort choices lead to different values of $\sum_{s_{t+1}} \eta(s_{t+1}) \pi(s_{t+1}, e_t(h_0, s_0, s^t))$. Further, for all (h_0, s_0, s^t) with $x_{ht}(h_0, s_0, s^t) = 0$, the corresponding effort choices must satisfy $e_t(h_0, s_0, s^t) \leq e^*$. This immediately follows from inequality (27) since $\sum_{s_{t+1}} \eta(s_{t+1}) \pi(s_{t+1}, e_t(h_0, s_0, s^t))$ is increasing in the effort choice e_t. In the next section, we show a stronger result: Effort levels are the same across all initial states and all histories, including the states and histories with $x_{ht}(h_0, s_0, s^t) = 0$.[7]

3.3. Optimal Allocations: Full Characterization

We continue to consider allocations $\{c_t, e_t, h_{t+1}, K_{t+1}\}$. Equation (27) defines one set of necessary conditions for optimal allocations. Another set of necessary conditions is provided by the inverse Euler equation (Golosov, Kocherlakota, and Tsvinski, 2003, Rogerson, 1995a). The inverse Euler equation also has to hold in any model with a saving technology including models with human capital investment (Stantcheva, 2017). In the current framework, this inverse Euler equation reads

$$c_t(h_0, s_0, s^t) \;=\; \left[\beta \left(1 + r_f(\tilde{K}(e^*))\right)\right]^{-1} \sum_{s_{t+1}} c_{t+1}(h_0, s_0, s^{t+1}) \pi(s_{t+1}, e_t(h_0, s_0, s^t)) \qquad (28)$$

[7] In the current setting with linear human capital investment technology, there are optimal allocations that display zero human capital investment for some initial states, (h_0, s_0), or some histories s^t. However, proposition 3 below shows that all optimal allocations are payoff-equivalent to an optimal allocation with $x_{ht}(h_0, s_0, s^t) > 0$ for all (h_0, s_0, s^t).

for all initial states, (h_0, s_0), and all histories s^t, where e^* is the effort level chosen in the case of $x_{ht}(h_0, s_0, s^t) > 0$. Equation (28) says that expected consumption growth is equal to $\beta(1 + r_f)$ for all (h_0, s_0, s^t). In other words, optimal individual consumption has the martingale property. The optimal individual consumption process follows a sub-martingale if $\beta(1 + r_f) > 0$, a martingale if $\beta(1 + r_f) = 0$, and a super-martingale if $\beta(1 + r_f) < 0$.

A direct implication of the martingale property (28) is that optimal individual consumption can be represented as

$$c_{t+1}(h_0, s_0, s^{t+1}) \;=\; \beta\left(1 + r_f(\tilde{K}(e^*)) + \epsilon_{t+1}(h_0, s_0, s^{t+1})\right) c_t(h_0, s_0, s^t) \tag{29}$$

where ϵ is a random variable that represents risk in individual consumption growth and has to satisfy

$$\sum_{s_{t+1}} \epsilon_{t+1}(h_0, s_0, s^{t+1})\pi(s_{t+1}, e_t(h_0, s_0, s^t)) \;=\; 0\ . \tag{30}$$

Equation (29) characterizing the consumption choice of the social planner is the analog to equation (23) describing the consumption choice of households in the market economy with taxes and transfers.[8]

Clearly, the choice of a consumption-effort allocation, $\{c_t, e_t\}$, is equivalent to the choice of a effort-risk allocation, $\{e_t, \epsilon_{t+1}\}$, together with a choice of initial consumption function, $c_0 = c_0(h_0, s_0)$. Suppose that the optimal allocation, $\{c_0, e_t, \epsilon_{t+1}\}$, is simple in the sense that $e_t(h_0, s_0, s^t) = e^*$ and $\epsilon_{t+1}(h_0, s_0, s^t, .) = \epsilon^*(.)$ for all (h_0, s_0, s^t). In this case, simple algebra using proposition 2 and the representation of optimal individual consumption (29) shows that the optimal effort-risk combination, (e^*, ϵ^*), together with the optimal capital-to-labor ratio, $\tilde{K}^*$, are the solution to the following static social planner problem:

$$\begin{gathered}
\max_{e,\epsilon,\tilde{K}} \left[-d(e) \;+\; \frac{\beta}{1-\beta}\sum_s ln\left(1 + r_f(\tilde{K}) + \epsilon(s)\right)\pi(s, e)\right] \\
subject\ to: \\
r_f(\tilde{K}) \;=\; \phi\, r_h(\tilde{K}) \;+\; \sum_s \eta(s)\pi(s, e)
\end{gathered} \tag{31}$$

---

[8] Note that taking the expectations over (h_0, s_0, s^t) in (29) shows that optimal aggregate consumption follows $C_{t+1} = \beta\left(1 + r_f(\tilde{K}(e^*))\right) C_t$. Thus, for given e^* and C_0, the optimal aggregate consumption path is pinned down.

$$\sum_s \epsilon(s)\pi(s,e) \;=\; 0 \;\; .$$

$$d'(e) \;=\; \frac{\beta}{1-\beta}\sum_s ln\left(1 + r_f(\tilde{K}) + \epsilon(s)\right)\frac{\partial \pi}{\partial e}(s,e)$$

The maximization problem (31) is the choice problem of a social planner who chooses effort level, e, consumption risk, ϵ, and a capital-to-labor ratio, $\tilde{K}$, so as to maximize welfare defined by the expected utility of households with log-utility function and consumption given by $\ln(1 + r_f(\tilde{K}) + \epsilon)$ subject to three constraints. The first constraint states that the return to financial capital investment is equal to the expected return to human capital investment, where the social planner can affect returns through the choice of the capital-to-labor ratio and the mean level of human capital shocks (effort). The second constraint says that ϵ is a variable representing risk and therefore has a fixed mean, which is normalized to zero. This last constraint is the analog of the requirement that transfer payments have to balance in the market economy – see equation (9). The final second constraint is the incentive compatibility constraint that ensures that for individual households will choose the prescribed effort choice.

The next proposition shows that optimal allocations are indeed simple, and can therefore be found by solving (31):

Proposition 3. Optimal allocations are simple. Specifically, let the triple $(e^*, \epsilon(.), \tilde{K}^*)$ be the solution to the static social planner problem (31) and assume at $(e^*, \tilde{K}^*)$ condition (21) holds. Then the optimal allocation is given by:

$$\begin{aligned}
e_t(h_0, s_0, s^t) &= e^* \qquad (32)\\
\epsilon_{t+1}(h_0, s_0, s^t, .) &= \epsilon^*(.)\\
\tilde{K}_{t+1} &= \tilde{K}^*\\
c_{t+1}(h_0, s_0, s^{t+1}) &= \beta\left(1 + r_f(\tilde{K}^*) + \epsilon^*(s_{t+1})\right) c_t(h_0, s_0, s^t)\\
c_0(h_0, s_0) &= (1-\beta)\left(1 + r_f(\tilde{K}_0)\right)\left(\tilde{K}_0 + 1/\phi\right) H_0 \, \frac{\mu(h_0, s_0)}{\pi_0(h_0, s_0)}
\end{aligned}$$

Further, one optimal allocation of individual human capital is[9]

$$h_{t+1}(h_0, s_0, s^t) \;=\; \beta(1 + r_f(\tilde{K}^*) + \epsilon^*(s_t))h_t(s^{t-1}) \; ,$$

[9]The optimal aggregate level of human capital investment, X_{ht}, is uniquely determined for all t. However, since the optimal effort choice, e^*, and therefore the optimal "depreciation rate" $\bar{\eta}(e^*)$, are common across

in which case optimal individual consumption can be represented as

$$c_t(h_0, s_0, s^t) \;=\; (1-\beta)\left(1 + r_f(\tilde{K}^*) + \epsilon^*(s_t)\right)\left(\tilde{K}^* + \frac{1}{\phi}\right) h_t(s^{t-1})\;.$$

A number of comments regarding proposition 3 are in order. First, proposition 3 implies that the cross-sectional distribution of consumption spreads out over time – the well-known immiseration result of Atkeson and Lucas (1992). If we introduce either stochastic death of households (Contantinides and Duffie, 1996) or a social welfare function that puts weight on future generations (Farhi and Werning, 2007, and Phelan, 2006) we can generate a stationary cross-sectional distribution of consumption while still keeping the tractability of the model.

Second, substituting the optimal consumption allocation into the lifetime utility function yields the lifetime utility for each household type associated with the optimal allocation:

$$\begin{aligned} U\left(\{c_t, e_t | h_0, s_0\}\right) \;&=\; \frac{1}{1-\beta}\left[\ln(1-\beta) \,+\, \frac{\beta}{1-\beta}\ln\beta \,+\, \ln\left((1 + r_0(\tilde{K}_0, \bar{\eta}_0) + \epsilon^*(s_0))(\tilde{K}_0 + 1/\phi)h_0\right)\right] \\ &\quad - \frac{d(e^*)}{1-\beta} \,+\, \frac{\beta}{(1-\beta)^2}\sum_s \ln(1 + r_f(\tilde{K}^*) + \epsilon^*(s))\pi(s, e^*) \end{aligned} \tag{33}$$

Representation (33) is the analog to the expression of lifetime utility in a market equilibrium (proposition 1). Further, optimal aggregate human capital and optimal aggregate consumption are obtained by taking the expectations in (32), and optimal aggregate physical capital is then determined through $K_t = \tilde{K}H_t$. If we define aggregate total wealth in period t as $W_t = K_t + \frac{H_t}{\phi}$, then this can be written as

$$\begin{aligned} C_t \;&=\; (1-\beta)\left(1 \,+\, r_f(\tilde{K}^*)\right) W_t \\ W_{t+1} \;&=\; \beta\left(1 \,+\, r_f(\tilde{K}^*)\right) W_t \\ K_t \;&=\; \frac{\phi\tilde{K}^*}{1+\phi\tilde{K}^*} W_t \;\;;\;\; H_t \;=\; \frac{1}{1+\phi\tilde{K}^*} W_t \end{aligned} \tag{34}$$

which is the analog of the (25) describing the equilibrium evolution of aggregate variables in the market economy.

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households, the optimal level of individual human capital investment is indeterminate. More specifically, any human capital allocation, $\{x_{ht}\}$, that is consistent with the optimal aggregate human capital path, $\{X_{ht}\}$, is optimal.

Third, in (31) we use the first-order conditions to characterize the optimal effort choice of individual households for given level of consumption risk. This is justified since in our setting first-order conditions are necessary and sufficient for the household effort choice as long as the standard assumptions for the one-period moral hazard problem are satisfied (Rogerson, 1985b). In general, first-order conditions might not be sufficient in repeated moral hazard problems with infinitely-lived agents – the product of two concave (probability) functions is not necessarily concave. Abaraham, Koehne, and Pavoni (2011) provide conditions in a two-period moral hazard problem that ensure necessity and sufficiency of first-order conditions.

3.4. Optimal Equilibrium Allocations

A comparison of the equilibrium allocation of a market economy (proposition 1) and the optimal allocation (proposition 3) shows the equivalence between the two when the tax- and transfer system is chosen appropriately. In addition, the welfare weights in the social planner problem have to be chosen in line with the distribution of initial wealth in the market economy. In the current setting, the welfare weights defined as

$$\mu(h_0, s_0) \quad = \quad \frac{(1 + r_0(\tilde{K}_0, \bar{\eta}_0) + \epsilon^*(s_0))h_0}{(1 + r_0(\tilde{K}_0, \bar{\eta}_0))H_0} \, \pi_0(h_0, s_0) \ , \tag{35}$$

will ensure that the c_0 chosen by the social planner is also the c_0 in the equilibrium of the market economy. More precisely, we have the following decentralization result:

Proposition 4. Suppose $(e^*, \epsilon^*, \tilde{K}^*)$ solves the static social planner problem (31) and condition (21) is satisfied. Define a tax-and-transfer system, (τ^*, tr^*), as the solution to

$$\begin{aligned}
\phi r_h(\tilde{K}^*)(1 + tr^*(s)) \quad &= \quad r_f(\tilde{K}^*) \, + \, (1 + \phi\tilde{K}^*)\epsilon^*(s) \, - \, \eta(s) \qquad (36)\\
0 \quad &= \quad \sum_s \frac{(1 - \tau_h^* + tr^*(s))\phi r_h(\tilde{K}^*) \, + \, \eta(s) - (1 - \tau_f^*)r_f(\tilde{K}^*)}{1 + r_f(\tilde{K}^*) + \epsilon^*(s)} \, \pi(s, e^*)\\
0 \quad &= \quad \phi\tilde{K}^* r_k(\tilde{K}^*) \, \tau_k^* \, + \, \phi r_h(\tilde{K}^*) \, \tau_h^*
\end{aligned}$$

Then $(e^*, \epsilon^*, \tilde{K}^*)$ and (τ^*, tr^*) define an optimal sequential market equilibrium.

The first equation in (36) ensures that transfer payments in the market economy are set so that social insurance is optimal. The condition is derived from a equalization of equilibrium consumption (23) and socially optimal consumption (29). The second equation in (36) states that taxes and subsidies have to be chosen so that the socially optimal portfolio allocation is an equilibrium outcome in the market economy. The last equation in (36) is the government budget constraint.

The following corollaries are straightforward implications of proposition 4:

Corollary 1. The optimal tax system requires a subsidy on human capital (risky) investment, $\tau_h^* < 0$, and a tax on physical capital (risk-free) investment, $\tau_a^* > 0$.

Corollary 2. Consider the set of all tax-and-transfer systems that are arbitrary functions of initial types and individual histories, $\tau_t = \tau_t(h_0, s_0, s^t)$ and $tr_t = tr_t(h_0, s_0, s^t)$, and the associated set of sequential market equilibria. The simple tax-and-transfer system specified in proposition 4 is the socially optimal in the sense that there is no tax-and-transfer system that leads to sequential market equilibria with higher social welfare.

Corollary 1 was first shown for private-information economies in Da Costa and Maestri (2007) using a one-period model with private information about types. The intuition underlying the result is simple. The optimality condition (27) requires that the expected return to human capital investment is equal to the risk-free rate. Since households are risk averse and human capital is risky, they can only be induced to invest in human capital if human capital investment is subsidized relative to investment in the risk-free asset.

The intuition underlying corollary 2 is straightforward. No tax- and transfer system can lead to equilibrium allocations that generate higher social welfare than the social welfare in an optimal allocation. Since the simple tax and transfer system defined in proposition 4 yields optimal social welfare it cannot be dominated by another tax and transfer system. Corollary 2 states this result in terms of arbitrary linear tax and transfer system, but the same result if we allow the government to use arbitrary non-linear tax and transfer systems.

3.5. A Rise in Human Capital Risk

We now consider an increase in human capital risk. It is standard (Rothschild and Stiglitz, 1970) to formalize the idea of an increase in risk by considering a situation in which the random variable, η', is a mean preserving spread of the random variable, η. In an economy with moral hazard and endogenous distribution of shocks, this definition is somewhat ambiguous. Specifically, even though the function describing the size of human capital shocks, $\eta = \eta(s)$, is an exogenous object, the underlying distribution of shocks, $\pi(., e)$, becomes an endogenous object in moral hazard economies with endogenous effort, e. In this paper, we model an increase in human capital risk as a change in the shock-function from $\eta = \eta(s)$ to $\eta' = \eta'(s)$ that is a mean-preserving spread if the same distribution of shocks, $\pi(., e^*)$, is used to define mean-preserving spread, where e^* is the equilibrium effort level in the original economy. This definition seems to capture best the notion of a change in fundamentals that keeps choices of agents fixed, which is the approach usually taken when discussing changes in the economic environment by using comparative statics analysis.

The following proposition is a straightforward implication of the equivalence between equilibrium allocations with optimal tax-and-transfer system and optimal allocations.

Proposition 5. A rise in human capital risk increases welfare if the tax-and-transfer system is adjusted optimally. More precisely, suppose human capital risk, η', is a mean-preserving spread of human capital risk, η, and consider the associated optimal equilibrium consumption-effort allocations, $\{c'_t, e'_t\}$ and $\{c_t, e_t\}$. Then we have for all households (h_0, s_0):

$$U(\{c'_t, e'_t | h_0, s_0\}) \quad \geq \quad U(\{c_t, e_t | h_0, s_0\})$$

The intuition underlying proposition 5 is straightforward. Optimal allocation are the solution to the static social planner problem (31), which is a simple one-agent decision problem. After the increase in human capital risk the original solution to (31) is still feasible so that social welfare cannot be reduced.

Note that this result does not hold when the tax-and-transfer system is held constant. In this case, the welfare effect is ambiguous. On the one hand, welfare is reduced due to more consumption risk. On the other hand, welfare increases because more opportunities are provided. See Heathcote, Storesletten, and Violante (2008) for an analysis of these two opposing effects in a standard macroeconomic model with incomplete markets and endogenous labor supply.

4. Calibrating the Model

In this section, we discuss the model specification and calibration for the quantitative analysis. We confine attention to a model with two shock realization and interpret the negative shock to human capital as the long-term earning loss of a displaced worker. Accordingly, we use the estimates of the empirical literature on job displacement risk to calibrate the human capital risk in the model economy.

4.1. Production

The basic time period is one year and the production technology is Cobb-Douglas: $Y = AK^{\alpha}H^{1-\alpha}$. Thus, the marginal product of physical capital and human capital, respectively, are given by:

$$\begin{aligned} F_k(\tilde{K}) &= \alpha A\tilde{K}^{\alpha-1} \\ F_h(\tilde{K}) &= (1-\alpha)A\tilde{K}^{\alpha}. \end{aligned} \tag{37}$$

Using $r_f = F_k - \delta_k$ and (31) we derive:

$$\delta_k = \alpha\left(\frac{K}{Y}\right) - r_f. \tag{38}$$

We choose $\alpha = 0.36$ for the income share of physical capital, a capital-to-output ratio $K/Y = 3$, and an annual real rate of return to physical capital $r_f = 0.06$. With these values equation (38) yields $\delta_k = 0.06$. We normalize $A = 1$ without loss of generality. Using $r_f = \alpha A\tilde{K}^{\alpha-1} - \delta_k$ implies a value $\tilde{K} = 5.5655$.

The equality-of-returns condition (27) can be written as:

$$\phi(1-\alpha)A\tilde{K}^{\alpha} + \overline{\eta}(e) = r_f \tag{39}$$

The variable $\overline{\eta}(e) = \sum_s \eta_s\pi_s(e)$ is the mean level of human capital changes not caused by goods investment in human capital, which is a combination of positive shocks (on the job learning-by-doing and improvement in match quality through job-to-job movements) and negative shocks (loss of human capital during unemployment spells, health shocks, retirement). For the calibration, we assume that these two opposing forces cancel each other out and set $\overline{\eta}(e) = 0$. Given the already assigned parameter values, equation (39) yields a value of ϕ, which we find to be $\phi = 0.0505$. The value of the implied portfolio share is $\theta = \frac{\phi\tilde{K}}{1+\phi\tilde{K}} = 0.2195$.

Note that denominating the stock if human capital in units of physical capital, the physical-to-human capital ratio is $\phi\tilde{K} = 0.2812$, a value roughly consistent with the empirical estimate by Liu (2011).

4.2. Preferences and Human Capital Risk

The growth rate of aggregate output and consumption is $g = \beta(1 + r_f) - 1$. Targeting a growth rate of $g = 0.0200$ yields a value for the discount factor of $\beta = 0.9623$.

The dis-utility function of effort is a power function:

$$d(e) = d_1 e^{d_2}. \tag{40}$$

We normalize $d_1 = 1$ without loss of generality and choose $d_2 = 2$. The value is consistent with the estimates used by Christensen et al. (2005) in their work on search unemployment who find $d_2 = 1.85$. Note that a quadratic dis-utility implies a value of the "Frisch-elasticity" of 1 which is between the higher value used in the macro-literature and the lower values typically obtained in microeconometric studies.

For the human capital risk, η, we focus on the event of job displacement and the associated loss of human capital. For simplicity, we assume that there are only two shock realizations: $\eta_s \in \{\eta_l, \eta_h\}$ with $\eta_l < \eta_h$. Given that we require $\bar{\eta}(e) = \eta_l\pi_l(e) + \eta_h\pi_h(e) = 0$ the process of job displacement risk is defined by the human capital loss in the case of job displacement, $\eta_l < 0$, and the probability of job displacement, $\pi_l(e)$. In the Appendix we show that the human capital shocks in the model economy correspond to permanent income shocks (log labor income follows a random walk) and we therefore set the size of the human capital shock in the case of job displacement, η_l, in order to match the (average) long-term earnings losses of displaced workers in the U.S. estimated by the empirical literature. The empirical literature discussed in the Appendix suggests that the long-term earnings losses of displaced workers in the U.S. are on average 15 percent and we use this value for our baseline calibration $(1 - \theta)\eta_l = 0.15$. As a result, we get $\eta_l = -0.1922$ and, because of $\bar{\eta}(e) = 0$, we get $\eta_h = 0.0080$.

We consider an effort technology that is an (adjusted) exponential distribution for given effort level:

$$\pi_h(e) = 1 - \lambda_1 \exp(-\lambda_2 e) - \lambda_3 \tag{41}$$

This leaves us with three free parameters of the effort technology, λ_1, λ_2, and λ_3. We choose the values of these parameters to match the following targets.

First, we require the job displacement risk of the calibrated model economy to be in line with the job displacement risk faced by U.S. workers. In the model economy, job displacement risk is defined by the probability of job displacement and the human capital loss associated with the displacement event. The empirical literature on job displacement in the U.S. is summarized in the Appendix. This literature suggest a value of 4 percent for the annual job displacement rate and we calibrate the model economy to match this target, i.e. $\pi_l(e) = \lambda_1 \exp(-\lambda_2 e) + \lambda_3 = 0.0400$.

Second, we confine attention to optimal equilibrium allocations, that is, we assume that the investment incentives provided by the U.S. tax system and the level of insurance against job displacement risk provided by the U.S. transfer system and the non-observable insurance through family and friends are socially optimal.[10] Specifically, we choose the tax parameters, τ_h and τ_f, as well as the transfer payments, tr_l and tr_h, to ensure that (36) holds (proposition 4). This leaves as with one free policy parameter since (36) defines only three equations, which we use to match the consumption drop upon job displacement estimated by the empirical literature.

Cochrane (1991) is one of the first empirical studies showing incomplete consumption insurance of US households against involuntary job loss. Subsequent studies that have directly focused on the negative consumption effects of job displacement have confirmed this finding. Specifically, Stephens (2001) estimates a long term decline (six years after displacement) in the earnings of the household head of 22 percent and a decline in family food consumption of half that amount (11 percent). In accordance with these estimates, we calibrate the model so that the consumption is about half of the long-term earnings loss associated with job displacement. More precisely, we set parameter values so that the solution of the social planner problem (31) satisfies $\epsilon_l = 0.15 - 0.07 = 0.08$ and set the insurance payments, tr_l, in the corresponding market equilibrium accordingly.

[10] There are three main sources of government insurance against the long-term losses associated with job displacement: Severance payments by firms, direct transfer payments by the government including payments for retraining programs, and the indirect insurance provided through the progressive nature of the tax system. See Parsons (2004) for a survey.

Third, we introduce one normalization. Specifically, we assume that without any effort, $e = 0$, job-displacement risk is one. Thus, (41) imposes a joint restriction on λ_1 and λ_3: $\pi_l(0) = \lambda_1 + \lambda_3 = 1$. Using all three restrictions we find the parameter values $\lambda_1 = 0.9643$, $\lambda_2 = 35.0120$, and $\lambda_3 = 0.0357$.

The values of all model parameters are summarized in Table 1.

Table 1. Parameter values for baseline calibration

parameter	description	value
α	income share of physical capital	0.3600
A	total factor productivity	1.0000
δ_k	depreciation rate of physical capital	0.0600
β	time preference factor	0.9623
d_1	dis-utility parameter 1	1.0000
d_2	dis-utility parameter 2	2.0000
λ_1	search technology parameter 1	0.9643
λ_2	search technology parameter 2	35.0120
λ_3	search technology parameter 3	0.0357
η_l	human capital loss (displacement shock)	-0.1922
η_h	human capital gain	0.0080

Note that we have calibrated the model to the constrained optimal allocation and targeted a transfer scheme tr accordingly match $\epsilon(s)$. Implementing the constrained optimal allocation also requires that the planner chooses capital and labor income taxes/subsidies to provide right incentives for the accumulation of physical and human capital. Specifically, we find a capital income tax rate τ_k of 0.23 percent and human capital tax rate (subsidy) τ_h of -0.13 percent, respectively. The difference $\tau_k - \tau_h = 0.36$, which is a measure of the net subsidy for human capital investment, is broadly in line with the empirical values for the U.S. (Trostel, 1993).

4.3. Job Displacement and Effort Choice

The empirical literature surveyed in the Appendix defines a displayed worker as an individual with established work history who is involuntarily separated from his job due to a mass layoff or plant closure. In contrast, other causes of job loss, such as quits or firings for cause, are not considered displacement (Kletzer, 1998). This definition of the empirical literature begs the question to what extent the moral hazard model analyzed in his paper, in which effort of individual workers affects the likelihood that a human capital loss occurs, provides an appropriate description of job displacement. There are two reasons why moral hazard is likely to be an important issue when it comes to job displacement and the corresponding job displacement risk estimated by the empirical literature.

First, work effort of individual employees and the resulting job performance is a crucial factor when employers use discretion whom to let go in the case of a mass lay off (Gibbons and Katz, 1991). Similarly, work effort and the resulting job performance is one determinant of the decision by employers whom to recall from a lay off, which is not counted as job displacement and is a very common event in the U.S. In both cases, the effort choice of workers determines the likelihood that the job displacement event, and therefore the human capital loss, occurs.

Second, effort choice also affects the size of the human capital loss associated with the displacement event. Specifically, work effort at the old (pre-displacement) job as well as search effort during the unemployment spell determine the match quality and corresponding pay at the new (post-displacement) job. As a simple example, consider the case in which a fraction $q(e)$ of the displaced workers experience no human capital loss because in the new job their human capital can be fully used and $(1 - q(e))$ suffer a human capital loss of η_l because only a fraction $(1 - \eta_l)$ of their human capital, h, can be usefully employed in the new job. If we denote the probability of job displacement by $\pi(e)$, then the probability that a human capital loss of size η_l occurs is equal to $(1 - q(e))\pi(e)$. Note that this example fits into our model with only two η-realizations if we set $\pi_l(e) = (1 - q(e))\pi(e)$.[11]

[11] Needless to say, a model with more than two η-realizations provides a more realistic description of this mechanism than the two-state model we use here.

There is (indirect) empirical support for a model in which effort choice affects the human capital loss associated with the displacement event. If we assume that a fraction $(1-\eta_l)$ of a worker's human capital is either sector-specific or occupation-specific, then human capital $(1-\eta_l)h$ is lost in the case that a displaced worker has two switch sectors or occupation to regain employment and search effort affect the likelihood that the switch does not have to occur (the unemployed worker receives a job offer in his old sector/occupation). Neal (1995) provides evidence that a substantial part of the long-term earnings losses of displaced workers is due to the loss of sector-specific human capital and Kambourov and Manovskii (2009a) show that occupation-specific human capital explains a significant portion of the long-term earnings losses of displaced workers. In addition, Gibbons and Katz (1991) find that workers displaced under slack work conditions, in which case employers have some discretion whom to lay off, experience longer jobless durations and lower post-displacement earnings than do workers laid off when a whole plant closes and selection possibilities are absent. Gibbons and Katz (1991) interpret their finding as evidence in favor of adverse selection, but moral hazard of the type discussed here can equally well explain their empirical result.

5. Quantitative Results

5.1. Increase in Human Capital Risk

We an increase in job displacement risk that is modelled as an increase in the spread of human capital shocks, η. Specifically, we assume that the human capital loss in the case of job displacement, η_l, increases so that the associated long-term earnings loss increases by 5 percentage points, that is, we set $\eta_l' = \eta_l - 0.05/(1-\theta)$. To keep the mean of the random variable η constant, we increase the human capital gain in the case that no job displacement occurs according to $\eta_h' = -\pi_l/\pi_h\eta_l'$, where we use the probabilities before the change in risk (constant effort) as in our theoretical analysis in proposition 5.[12]

The increase in earnings losses of displaced workers of 5 percentage points is motivated

[12]Our approach to modeling an increase in labor market risk is very similar to the approach taken in Ljungqvist and Sargent (1998, 2008), who also focus on the event of job loss and consider an increase in the size of the associated human capital loss. In contrast, Krueger and Perri (2006)and Heathcote, Storesletten, and Violante (2010) study labor market risk in its totality and consider an increase in the variance of the change in labor income as estimated, for example, by Gottschalk and Mofitt (1994).

by two pieces of evidence. First, in line with our calibration, Kambourov and Manovskii (2009a) find that displaced workers in the US suffer on average a 15 percent loss in weekly earnings five year preceding the displacement event. However, those workers who stay in the same occupation experience a reduction in weekly earnings of only 6 percent, whereas workers who switch their occupation experience a loss in weekly earnings of 18 percent. This finding suggests that 3/4 of the average earnings losses of displaced workers is due the loss of occupation-specific human capital.

Second, Kambaurov and Manovskii (2008) use PSID data and find that the average level of occupational mobility in the US has increased over the 1968-1997 period from 10% to 15% at the one-digit level, 12% to 17% at the two-digit level, and 16% to 20% at the three digit level. This suggest an increase of occupation mobility by about 40 percent, which implies that on average the loss of occupation-specific capital associated with job displacement has increased by $0.75 \times 0.4 = 0.30\%$ over the period $1968 - 1997$. For our calibrated model economy, this translates into an increase of the long-term earnings losses of displacement from 15 percent to 19.5 percent, which we round up to an increase of 5 percentage points.[13]

5.2. Results

We decompose the total welfare effect of increasing human capital risk into three components: A welfare effect associated with changes in effort, a welfare effect associated with the cost of risk due to risk aversion, and a welfare effect that is due to a change in the growth rate of consumption/output. Accordingly, we define the value functions and their decompositions as follows:

$$\begin{aligned}
(1-\beta)V &= -d(e) + \frac{\beta}{1-\beta}\sum_s \ln(1 + r_f(\tilde{K} + \epsilon(s)))\pi(s,e) \qquad (42)\\
(1-\beta)V_e &= -d(e)\\
(1-\beta)V_g &= \frac{\beta}{1-\beta}\ln\left(\sum_s (1 + r_f(\tilde{K} + \epsilon(s)))\pi(s,e)\right)
\end{aligned}$$

[13]Neal (1995) argues that a substantial part of the estimated earnings losses of displaced workers are due to the loss of industry-specific human capital. Kambaurov and Manovskii (2008) show that industry mobility has been rising as well in the U.S. over the period 1968-1997. Our analysis would equally apply if the earnings losses of displaced workers are dues to the loss of industry-specific human capital.

$$(1-\beta)V_r = \frac{\beta}{1-\beta}\sum_s \ln(1+r_f(\tilde{K}+\epsilon(s))\pi(s,e) - \frac{\beta}{1-\beta}\ln\left(\sum_s (1+r_f(\tilde{K}+\epsilon(s))\pi(s,e))\right).$$

Note that we have $V = V_e + V_g + V_r$. We further define the corresponding concepts of the welfare cost/gains in consumption equivalent units as

$$\begin{aligned} \Delta &= \exp((1-\beta)(V' - V)) - 1 \qquad (43) \\ \Delta_e &= \exp((1-\beta)(V'_e - V_e)) - 1 \\ \Delta_g &= \exp((1-\beta)(V'_g - V_g)) - 1 \\ \Delta_r &= \exp((1-\beta)(V'_r - V_r)) - 1 \end{aligned}$$

Note that in general $\Delta \neq \Delta_e + \Delta_g + \Delta_r$, but it turns out that in our quantitative analysis the equality holds approximately: $\Delta \approx \Delta_e + \Delta_g + \Delta_r$.

Table 2. Welfare Effects of an Increase in Job Displacement Risk

variable	optimal policy response	no policy response
Δ (welfare effect)	0.0358	- 0.1673
Δ_e (effort effect)	- 0.1192	- 0.3964
Δ_g (growth effect)	0.2650	0.7388
Δ_r (risk effect)	- 0.1104	- 0.5144

Note: Welfare effect Δ and its decmosition Δ_e,Δ_g,and Δ_r are in percent of consumption equivalent units. Scenario "no adjustment" refers to a situation in which the transfer scheme remains unaffected but the capital income tax rate is adjusted to balance the government budget.

There are two main results. First, the optimal policy response to the rise in η_l by 0.05 is to increase social insurance, tr_l, substantially so that the consumption drop, η_l, only increases by 0.013 from 0.0800 to 0.0931 – only one fourth of the initial rise in job displacement risk is passed on into consumption. As a consequence, the net effect on work effort and welfare is rather modest. Specifically, the probability of job displacement, π_i, decreases by $1,25$ percent from $0,04$ to $0,0395$ and welfare increases by $0,036$ percent of lifetime consumption.

Our second result is that the social welfare cost of not adjusting the social insurance system is substantial. Specifically, keeping the generosity of the social insurance system fixed, the observed rise in job displacement risk leads to a substantial increase in the consumption loss of displaced workers. Specifically, the increase of η_l by 0.05 results in an increase in the consumption drop by roughly 0.035 – about two thirds of the increase in job displacement risk is passed on into consumption risk. Consequently, the effort response of individual households is significant – the probability of job displacement, π_i, decreases by $3,75$ percent from $0,04$ to $0,0385$. Finally, the welfare loss of not adjusting the social insurance system is equal to $0.1673 - (-0.0358) = 0.2031$ of lifetime consumption – a substantial loss. [14]

[14] A further result is that in both case, optimal policy response and no policy response, the change in portfolio holdings, θ, is small producing negligible welfare effects.

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Appendix

Proof of Proposition 1.

We begin with the proof that the household plan specified in proposition 1 solves the sequential household maximization problem (5). To this end, use the change of variables (16) to define a new sequential household maximization problem with plans $\{c_t, e_t, w_{t+1}, \theta_{t+1}|w_0, \theta_0, s_0\}$ as choice variables. Further, let us modify the household maximization problem and replace the non-negativity constraint on human capital investment, $x_{ht} \geq 0$, by the constraint that human capital has to be non-negative, which reads: $1 - \theta_{t+1} \geq 0$. Clearly, the choice set in the utility maximization problem (5) is a subset of the choice set associated with this new household maximization problem. Thus, any solution to the new household maximization problem that satisfies the non-negativity constraint on human capital investment is also a solution to the original household maximization problem.

The Bellman equation associated with the new household maximization problem reads

$$\begin{aligned} V(w,\theta,s) \quad &= \quad \max_{c,e,w',\theta'} \left\{ \ln c \,-\, d(e) \,+\, \beta \sum_{s'} V(w',\theta',s')\pi(s',e) \right\} \qquad \text{(A1)} \\ s.t. \quad & w' \,=\, (1 + r(\theta,s))w - c \\ & w' \geq 0 \ \ ; \ \ \theta' \leq 1 \end{aligned}$$

Guess-and-verify shows that the household policy function specified in proposition 1 solves the Bellman equation (A1). Thus, by the principle of optimality the plan generated by this policy function solves the corresponding sequential household maximization problem. Given that the non-negativity constraint $x_{ht} \geq 0$ holds by assumption, this plan is also the solution to the original sequential household maximization problem (5).

There are two technical issues regarding the principle of optimality. First, the Bellman equation (A1) and the associated sequential household maximization problem have the property that probabilities depend on choices. In contrast, in the class of maximization problems analyzed in Stokey and Lucas (1989), probabilities do not depend on choices made by the decision maker. However, it is straightforward to show that the standard argument for the principle of optimality still applies in the extension when probabilities depend on choices. Similarly, another standard argument shows that the Bellman equation (A1) has a unique solution in an appropriately defined function space (contraction mapping theorem).

The second issue is the question of the construction of the appropriate function space since the economic problem is naturally an unbounded problem. To deal with this issue, one can, for example, follow Streufert (1990) and consider the set of continuous functions $\mathbf{B_W}$ that are bounded in the weighted sup-norm $\|V\| \doteq \sup_x \frac{|V(x)|}{W(x)}$, where $x = (w, \theta, s)$ and the weighting function W is given by $W(x) = |L(x)| + |U(x)|$ with U an upper bound and L a lower bound, and endow this function space with the corresponding metric. In other words, $\mathbf{B_W}$ is the set of all functions, V , with $L(x) \leq V(x) \leq U(x)$ for all $x \in \mathbf{X}$. A straightforward but tedious argument shows that confining attention to this function space is without loss of generality. More precisely, one can show that there exist functions L and H so that for all candidate solutions, V , we have $L(x) \leq V(x) \leq H(x)$ for all $x \in \mathbf{X}$.[15]

It remains to be shown that the intensive-from market clearing $\tilde{K} = \frac{\theta}{\phi(1-\theta)}$ implies the market clearing condition (8) and that the government budget constraint (9) reduces to the condition (25). This is shown by substituting the households policy function (18) into the aggregate conditions (8) and (9).

Proof of Proposition 2.

We prove proposition 2 in four steps. To ease the notation, we suppress the dependence of plans on (h_0, s_0).

Step 1. Existence of solution

According to the Weierstrass Theorem it suffices to show that the objective function in the maximization problem (14) is upper semi-continuous and the constraint set is compact. Using a variant of the arguments made in Becker and Boyd (1997), a straightforward argument shows that both properties hold if we choose the product topology to define the underlying metric space.

Step 2. Equality of returns (26) holds if $x_{ht}(s^t) > 0$.

We now prove that (26) is a necessary condition for optimal allocations if $x_{ht}(s^t) > 0$. Clearly, a straightforward approach to deriving the necessity of condition (26) is to write down the Lagrangian associated with the social planner problem and then to take first-order conditions. However, the existence of a vector of Lagrange multipliers requires additional

[15] Alvarez and Stokey (1998) provide a different, but related, argument to prove the existence and uniqueness of a solution to the Bellman equation for a class of unbounded problems similar to the one considered here, though without moral hazard.

conditions that might not be satisfied.[16] We therefore use a direct approach that does not require any further assumptions on the primitives.

To prove the claim, suppose not, that is, for the optimal allocation $\{c_t, e_t, k_t, h_t\}$ there exists a $\bar{t}$ and $\bar{s}^{\bar{t}}$ so that $x_{h\bar{t}}(\bar{s}^{\bar{t}})$ with $x_{h\bar{t}}(\bar{s}^{\bar{t}}) > 0$ and (26) is not satisfied:

$$\phi\, F_h(\tilde{K}_{\bar{t}+1}) \;+\; \sum_{s_{\bar{t}+1}} \eta(s_{\bar{t}+1})\pi(s_{\bar{t}+1}|e_t(\bar{s}^{\bar{t}})) \quad > \quad F_k(\tilde{K}_{\bar{t}+1}) - \delta_k \; . \tag{A2}$$

Inequality (A2) states that the expected value of human capital returns (the left-hand-side of A2) exceeds the risk-free return on physical capital investment (the right-hand-side of A2). The proof by contradiction for the reversed case is, mutatis mutandis, the same.

Consider an alternative allocation $\{\hat{c}_t, e_t, \hat{k}_t, \hat{h}_t\}$ with identical $\{e_t\}$ and a $\{\hat{c}_t, \hat{k}_t, \hat{h}_t\}$ that only differs from $\{c_t, k_t, h_t\}$ at history $\bar{s}^{\bar{t}}$ and for all $s_{\bar{t}+1}$ subsequent to $\bar{s}^{\bar{t}}$. More specifically, we define

$$\begin{aligned} \hat{h}_{\bar{t}+1}(\bar{s}^{\bar{t}}) \;&=\; h_{\bar{t}+1}(\bar{s}^{\bar{t}}) \;+\; (1+\eta(s_t))h_t + \phi\,(x_{ht} + \Delta x) \qquad\qquad \text{(A3)} \\ \hat{k}_{\bar{t}+1}(\bar{s}^{\bar{t}}) \;&=\; k_{\bar{t}+1}(\bar{s}^{\bar{t}}) - \Delta x \\ \forall s_{\bar{t}+1} \;:\; \hat{c}_{\bar{t}+1}(\bar{s}^{\bar{t}}, s_{\bar{t}+1}) \;&=\; c_{\bar{t}+1}(\bar{s}^{\bar{t}}, s_{\bar{t}+1}) + \Delta c(s_{\bar{t}+1}) \; , \end{aligned}$$

where the changes $\Delta x > 0$ and $\Delta c(s_{\bar{t}+1}) > 0$ are strictly positive real numbers. In words: in period $\bar{t}$, the alternative allocation increases human capital investment by Δx and reduces physical capital investment by Δx for households of type $\bar{s}^{\bar{t}}$, and in period $\bar{t}+1$ it increases consumption for these households in all possible states. Clearly, this allocation strictly increases social welfare. We now show that such as a strictly positive vector $(\Delta x, \vec{\Delta c})$ exists so that $\{\hat{c}_t, e_t, \hat{k}_t, \hat{h}_t\}$ satisfies the aggregate resource constraint and the incentive constraint, which contradicts the claim that $\{c_t, e_t, k_t, h_t\}$ is an optimal allocation. The idea of the proof is to show that the investment change increases available resources in $\bar{t}+1$ for small enough Δx and that the additional resources can be used to increase consumption in each state $s_{\bar{t}+1}$ without affecting the incentive constraint.

Since F is continuously differentiable the increase in human capital investment in period $\bar{t}$ by Δx increases production in period $\bar{t}+1$ by

$$\phi F_{h,\bar{t}+1}\Delta x \;+\; \epsilon_1(\Delta x) \tag{A4}$$

[16]See Rustichini (1998) for a general treatment of the question of the existence of a Lagrange vector in infinite-dimensional optimization problems with incentive constraints.

with $\lim_{\Delta x \to 0} \frac{\epsilon_1(\Delta x)}{\Delta x} = 0$. To reverse the increase in human capital investment in period $\bar{t}$, in the alternative allocation investment in human capital in period $\bar{t}+1$ is reduced by $\Delta x'(s_{\bar{t}+1})$. Since we require $\hat{h}_{\bar{t}+2} = h_{\bar{t}+2}$, the two investment changes Δx and $\Delta x'$ need to satisfy

$$\Delta x'(s_{\bar{t}+1}) \; = \; (1 + \eta(s_{\bar{t}+1}))\Delta \tag{A5}$$

Finally, the reduction in investment in physical capital in period $\bar{t}$ by Δx reduces output by $(F_{k,\bar{t}+1} - \delta_k)\,\Delta x \, + \, \epsilon_2(\Delta x)$ and the increase in physical capital investment in period $\bar{t}+1$ by Δx necessary to achieve $\hat{k}_{\bar{t}+2}(\bar{s}^{\bar{t}}, s_{\bar{t}+1}) = k_{\bar{t}+2}(\bar{s}^{\bar{t}}, s_{\bar{t}+1})$ reduces available resources in period $\bar{t}+1$ by $\Delta x + \epsilon_3(\Delta x)$, where $\lim_{\Delta x \to 0} \frac{\epsilon_2(\Delta x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\epsilon_3(\Delta x)}{\Delta x} = 0$.

In sum, for the alternative allocation $\{\hat{c}_t, e_t, \hat{k}_t, \hat{h}_t\}$ the additional resources available for consumption in period $\bar{t}+1$ for households of type $\bar{s}^{\bar{t}}$ are

$$\begin{aligned} \Delta\omega \;\; &= \;\; \phi F_{h,\bar{t}+1}\,\Delta x \\ &+ \;\; \left(1 + \sum_{s_{\bar{t}+1}} \eta(s_{\bar{t}+1})\pi(s_{\bar{t}+1}|e_{\bar{t}}(\bar{s}^{\bar{t}})\right)\Delta x \\ &- \;\; (1 + F_{1,\bar{t}+1} - \delta_k)\,\Delta x \, + \, \epsilon(\Delta x) \end{aligned} \tag{A6}$$

with $\lim_{\Delta x \to 0} \frac{\epsilon(\Delta x)}{\Delta x} = 0$. Using the assumption that expected human capital returns exceed the financial returns, we conclude that for small enough Δx we have $\Delta\omega > 0$.

Take a strictly positive real number Δu and define $\Delta c(s_{\bar{t}+1})$, for each $s_{\bar{t}+1}$, as the solution to

$$\ln\left(c_{\bar{t}+1}(\bar{s}^{\bar{t}}) + \Delta c(s_{\bar{t}+1})\right) \;\; = \;\; \ln\left(c_{\bar{t}+1}(\bar{s}^{\bar{t}})\right) \, + \, \Delta u \tag{A7}$$

Since the logarithmic function is continuous and strictly increasing in c we can always find positive real numbers $\Delta c(s_{\bar{t}+1})$ so that (A7) holds for given Δu. Further, continuous differentiability of the logarithmic function implies for sufficiently small Δu that the solution $\Delta\vec{c}$ to (A7) satisfies $\sum_{s_{\bar{t}+1}} \Delta c(s_{\bar{t}+1})\pi(s_{\bar{t}+1}|e_t(\bar{s}^{\bar{t}})) \;=\; \Delta\omega$. Thus, the alternative allocation $\{\hat{c}_t, e_t, \hat{k}_t, \hat{h}_t\}$ satisfies the aggregate resource constraint. It also satisfies the incentive constraint since

$$\begin{aligned} \sum_{s_{\bar{t}+1}} \ln\left(\hat{c}_{\bar{t}+1}(\bar{s}^{\bar{t}})\right)\pi(s_{\bar{t}+1}|e_t(\bar{s}^{\bar{t}})) \;\; &= \;\; \sum_{s_{\bar{t}+1}} \ln\left(c_{\bar{t}+1}(\bar{s}^{\bar{t}}) + \Delta c(s_{\bar{t}+1})\right)\pi(s_{\bar{t}+1}|e_t(\bar{s}^{\bar{t}})) \\ & \;\;\;\;\; \sum_{s_{\bar{t}+1}} \ln\left(c_{\bar{t}+1}(\bar{s}^{\bar{t}})\pi(s_{\bar{t}+1}|e_t(\bar{s}^{\bar{t}}))\right) \, + \, \Delta u \end{aligned} \tag{A8}$$

for any probability distribution π over states $s_{\bar{t}+1}$. This completes the proof of step 2.

Step 3. Constant effort choice for histories with $x_{ht}(s^t) > 0$ and constant $\tilde{K}$.

To see that effort choices are constant across histories with $x_{ht}(s^t) > 0$, consider the equality-of-expected-returns condition (26). This equation immediately implies that effort choices cannot depend on histories, $e_t(s^t) = e_t$, since we assume that higher effort increases the expected success – different effort choices lead to different values of $\sum_{s_{t+1}} \eta(s_{t+1})\pi(s_{t+1}, e_t(s^t))$.

A standard argument by contradiction also shows that e_t and $\tilde{K}_{t+1}$ do not depend on t. To prove this time-independence, it is crucial that the production function has constant-returns-to-scale with respect to K and H and that both production factors can be instantaneously adjusted at no cost.

Step 4. The case $x_{ht}(s^t) = 0$.

Consider now $x_{ht}(\bar{s}^t) = 0$ for some $\bar{s}^t$. In this case, we can repeat the contradiction argument made in step 2 that the inequality (A2) cannot hold. However, the reverse inequality cannot be ruled because the contradiction argument requires to reduce $x_{ht}(\bar{s}^t)$. Thus, we can conclude that inequality (2) holds for all households, but this inequality cannot be sharpened to an equality if $x_{ht}(\bar{s}^t) = 0$. This completes the proof of proposition 2.

Proof of Proposition 3.

The proof is conducted in five steps.

Step 1 Consumption implications of the Inverse Euler equation

Proof. For economies without human capital investment, Fahri and Werning (2012), Golosov, Kocherlakota, and Tsvinski (2003) or Rogerson (1995a) show that any optimal allocation with $X_{kt} > 0$ has to satisfy the inverse Euler equation (28). The proof only requires that aggregate consumption can be shifted across periods through adjustments in physical capital investment, which means that the inverse Euler equation (28) is also a necessary condition for optimal allocation when human capital is a choice variable. Stantcheva (2017) contains an explicit proof of the necessity of the Euler equation in economies with human capital investment. Note that equation (28) has to hold for all initial states (h_0, s_0) and all histories s^t, including initial states and histories with $x_{ht}(h_0, s_0, s^t) = 0$.

A direct implication of the martingale property (28) is that optimal individual consumption can be represented as

$$c_{t+1}(h_0, s_0, s^{t+1}) \;=\; \beta\left(1 + r_f(\tilde{K}(e^*)) + \epsilon_{t+1}(h_0, s_0, , s^{t+1})\right) c_t(h_0, s_0, s^t) \tag{A9}$$

where ϵ is a random variable that represents risk in individual consumption growth and has to satisfy

$$\sum_{s_{t+1}} \epsilon_{t+1}(h_0, s_0, s^{t+1})\pi(s_{t+1}, e_t(h_0, s_0, s^t)) \;=\; 0\ . \tag{A10}$$

Solving equation (A9) backward yields the following representation of individual optimal consumption:

$$c_t(h_0, s_0, s^t) \;=\; c_0(h_0, s_0)\prod_{n=1}^{t}\left(1 + r_f(\tilde{K}(e^*)) + \epsilon_n(h_0, s_0, s^n)\right) \tag{A11}$$

Taking the expectations over (A9) and (A11) shows that optimal aggregate consumption grows at rate $\beta(1 + r_f)$ and that the optimal path of aggregate consumption is pinned down once e^* and C_0 are determined:

$$\begin{aligned} C_{t+1} &= \beta\left(1 + r_f(\tilde{K}(e^*))\right) C_t \\ C_t &= \left(1 + r_f(\tilde{K}(e^*))\right)^t C_0 \end{aligned} \tag{A12}$$

Clearly, the social planner problem of choosing an allocation $\{c_t, e_t, h_{t+1}, K_{t+1}\} \in \mathbf{A}$ to maximize social welfare in (14) is equivalent to the social planner problem of choosing an allocation $\{c_0, \epsilon_{t+1}, e_t, h_{t+1}, K_{t+1}\} \in \tilde{\mathbf{A}}$ to maximize the social welfare function

$$\sum_{h_0, s_0}\left[\frac{1}{1-\beta}\ln c_0(h_0, s_0) \;+\; \tilde{U}(\{\epsilon_t, e_t | h_0, s_0\})\right]\mu(h_0, s_0) \tag{A13}$$

with

$$\begin{aligned} \tilde{U}\left(\{\epsilon_t, e_t | h_0, s_0\}\right) \;\doteq\; & -\sum_{t=0}^{\infty}\beta^t\sum_{s^t} d(e_t(h_0, s_0, s^t))\pi_t(s^t | e^{t-1}(h_0, s_0, s^{t-1})) \\ & + \frac{1}{1-\beta}\sum_{t=1}^{\infty}\beta^t\sum_{s^t}\ln(1 + r_f(\tilde{K}(e^*)) + \epsilon_t(h_0, s_0, s^t))\pi_t(s^t | e^{t-1}(h_0, s_0, s^{t-1})) \end{aligned} \tag{A14}$$

and a constraint set $\tilde{\mathbf{A}}$ that is defined, mutatis mutandis, in the same way as the constraint set $\mathbf{A}$. Note that the incentive constraint (12) now reads

$$\forall(h_0, s_0, s^t), \forall t\,,\ \forall\{\hat{e}_{t+n} | h_0, s_0, s^t\}\ :\ \tilde{U}_t(\{\epsilon_{t+n}, e_{t+n} | h_0, s_0, s^t\}) \;\geq\; \tilde{U}_t(\{\epsilon_{t+n}, \hat{e}_{t+n} | h_0, s_0, s^t\}) \tag{A15}$$

where $\{\hat{e}_{t+n}|h_0, s_0, s^t\}$ denotes the continuation plan and $\tilde{U}_t$ the continuation utility.

Note that even if the inverse Euler equation des not hold, any consumption-effort allocation, $\{c_t, e_t\}$, can be represented as an allocation $\{c_0, \epsilon_{t+1}, e_t\}$ using (A9) to define $\{\epsilon_{t+1}\}$. Thus, we can always introduce a change of variables so that the original social planner problem can be represented in terms of choosing an allocation $\{c_0, \epsilon_{t+1}, e_t, h_{t+1}, K_{t+1}\}$. However, in general (A10) does not have to hold sequentially, that is, for all (h_0, s_0, s^t). Property (A10) plays a crucial role in our proof of steps 3 and 4 below, and the inverse Euler equation ensures that the optimal $\{\epsilon_{t+1}, e_t\}$ satisfies condition (A10).

Step 2 In period $t = 0$, optimal consumption is $c(h_0, s_0) \;=\; C_0 \frac{\mu(h_0, s_0}{\pi(h_0,s_0)}$

Proof. The structure of the new social planner problem defined in step 1 implies that the optimal $c_0(.)$ has to solve

$$\begin{aligned} \max_{c_0(.)} & \sum_{h_0,s_0} \ln c_0(h_0, s_0)\mu(h_0, s_0) \\ s.t. & \sum_{h_0,s_0} c_0(h_0, s_0)\pi(h_0, s_0) \;=\; C_0 \end{aligned} \tag{A16}$$

where C_0 is aggregate consumption in period $t = 0$. Clearly, the solution to (A16) is

$$c(h_0, s_0) \;=\; C_0 \frac{\mu(h_0, s_0)}{\pi(h_0, s_0)} \;. \tag{A17}$$

Step 3 At $t = 0$, we have for all (h_0, s_0)

$$\begin{aligned} e_0(h_0, s_0) &= e^* \\ \epsilon_1(h_0, s_0, .) &= \epsilon^*(.) \end{aligned} \tag{A18}$$

where the function $\epsilon^*(.)$ is the solution to

$$\begin{aligned} \max\nolimits_{\epsilon(.)} \quad & \left[-d(e^*) \;+\; \frac{\beta}{1-\beta} \sum_s \ln(1 + r_f(\tilde{K}(e^*)) + \epsilon(s))\pi(s, e^*) \right] \\ s.t. \quad & \sum_s \epsilon(s)\pi(s, e^*) \;=\; 0 \\ & d'(e^*) \;=\; \frac{\beta}{1-\beta} \sum_s \ln(1 + r_f(\tilde{K}(e^*)) + \epsilon(s)) \frac{\partial \pi}{\partial e}(s, e^*) \end{aligned} \tag{A19}$$

Proof. We prove the claim by contradiction. To this end, suppose that there is an optimal allocation with $e_0(\bar{h}_0, \bar{s}_0) = \bar{e} < e^*$ for some $(\bar{h}_0, \bar{s}_0)$. Consider an alternative allocation that is identical to the original allocation except that $e_0(\bar{h}_0, \bar{s}_0)$ is increased and X_{h0} is simultaneously decreased so that the available aggregate resources in period 1 remain unchanged. Further, the resources freed up in period 0 through the reduction in X_{h0} are used to increase c_0 at $(\bar{h}_0, \bar{s}_0)$, where $\epsilon_1(.)$ is changed in order to ensure incentive compatibility. The new allocation is resource-feasible by construction. In addition, we will argue below that this new allocation is incentive-compatible manner and increases the lifetime utility for the household $(\bar{h}_0, \bar{s}_0)$. Since it does not change the expected lifetime utility of any other household $(h_0, s_0) \neq (\bar{h}_0, \bar{s}_0)$ it contradicts the claim that the original allocation with $e_0(\bar{h}_0, \bar{s}_0) = \bar{e} < e^*$ was socially optimal.

Clearly, the increase in c_0 at $(\bar{h}_0, \bar{s}_0)$ increase the lifetime utility of household $(\bar{h}_0, \bar{s}_0)$. Indeed, one can show that the increase in lifetime utility is at least as large as

$$\frac{\left[\sum_{s_1}(1+\eta(s_1))\frac{\partial \pi}{\partial e}(s_1, \bar{e})\right](1+\eta(\bar{s}_0))\bar{h}_0}{\phi \sum_{s_1}(1+\eta(s_1))\pi(s_1, \bar{e})} \times \left[(1-\beta)c_0(\bar{h}_0, \bar{s}_0)\right]^{-1} , \tag{A20}$$

but the particular value of the increase will not matter for our argument. There is also a utility cost associated with the move to a new allocation with higher effort level at $(\bar{h}_0, \bar{s}_0)$. To compute this utility cost, define the value function, v, as

$$\begin{aligned} \forall e_0 : \; v(e_0) = \;\; & \max_{\epsilon_1(.)} \left[-d(e_0) + \frac{\beta}{1-\beta}\sum_{s_1} \ln(1 + r_f(\tilde{K}(e^*)) + \epsilon_1(s_1))\pi(s_1, e_0)\right] \\ s.t. \quad & \sum_{s_1} \epsilon_1(s_1)\pi(s_1, e_0) \; = \; 0 \\ & d'(e_0) \; = \; \frac{\beta}{1-\beta}\sum_{s_1} \ln(1 + r_f(\tilde{K}(e^*)) + \epsilon_1(s_1))\frac{\partial \pi}{\partial e_0}(s_1, e_0) \end{aligned} \tag{A21}$$

Below we show that to choose ϵ_1 according to (A21) ensures incentive compatibility since $\epsilon_1(.)$ is only part of an optimal allocation if it solves the maximization problem in (A21). Thus, the utility cost of increasing e_0 and simultaneously adjusting ϵ_1 to ensure incentive compatibility is given by the derivative of v. By the envelope theorem, this derivative is zero:

$$v'(e_0) \; = \; 0 \tag{A22}$$

Hence, we have a contradiction since the increase in $e_0(\bar{h}_0, \bar{s}_0) < e^*$ increases utility by making an increase in c_0 possible without creating a utility cost.

It remains to be shown that an optimal $\epsilon_1(.)$ has to solve (A21). To prove this, suppose not, that is, $\epsilon_1(.)$ is part of an optimal allocation, but it does not solve (A21) for some $(\bar{h}_0, \bar{s}_0)$ and corresponding $\bar{e}_0 = e_0(\bar{h}_0, \bar{s}_0)$. Consider an alternative allocation identical to the original allocation with the exception that $\epsilon_1(.)$ is replaced by $\epsilon_1'(.)$, where $\epsilon_1'(.)$ is the solution to (A21). Clearly, the new allocation increases the lifetime utility of household $(\bar{h}_0, \bar{s}_0)$ and leaves the lifetime utility of all other households unchanged – it therefore increases social welfare. In addition, it satisfies the incentive constraint (A15), which is the desired contradiction. To see that the new allocation satisfies the incentive constraint, note that the set of constraints (A15) has only been altered at $(\bar{h}_0, \bar{s}_0)$. Further, since the incentive constraint holds for the original allocation, the lifetime utility function $\tilde{U}$ is additive, and the new allocation differs from the old allocation only with respect to $\epsilon_1(.)$, it suffices that $\bar{\epsilon}_1(.)$ satisfies

$$\begin{aligned} \forall\, \hat{e}_0 : \quad & -d(\bar{e}_0) + \tfrac{\beta}{1-\beta} \textstyle\sum_s ln \left(1 + r_f(\tilde{K}(e^*)) + \epsilon_1'(s_1)\right) \pi(s_1, \bar{e}_0) \\ & \geq -d(\hat{e}_0) + \tfrac{\beta}{1-\beta} \textstyle\sum_s ln \left(1 + r_f(\tilde{K}(e^*)) + \epsilon_1'(s_1)\right) \pi(s_1, \hat{e}_0) \end{aligned} \tag{A23}$$

In other words, for given $\epsilon_1'(.)$ the effort level $\bar{e}_0$ has to maximize the function

$$-d(e_0) + \frac{\beta}{1-\beta} \sum_s ln \left(1 + r_f(\tilde{K}(e^*)) + \epsilon_1'(s_1)\right) \pi(s_1, e_0) \tag{A24}$$

Given our assumption on d and π, first-order conditions are necessary and sufficient for this maximization problem (Rogerson, 1985). Differentiation of the objective function (A24) yields the last constraint in the maximization problem (A21), which shows that the new allocation satisfies the incentive constraint and completes the proof of step 3.

Step 4 For all (h_0, s_0, s^t) and all $t = 1, \ldots$ we have

$$\begin{aligned} e_t(h_0, s_0, s^t) &= e^* \\ \epsilon_{t+1}(h_0, s_0, s^t, .) &= \epsilon^*(.) \end{aligned} \tag{A25}$$

where $\epsilon^*(.)$ is the solution to (A19).

Proof. We prove the claim by induction. For $t = 0$ the claim has been proved in step 3. To prove the induction step from t to $t+1$, assume that the claim holds for t, that is, for all (h_0, s_0, s^t) we have (A25). We will prove that this implies that

$$\begin{aligned} e_{t+1}(h_0, s_0, s^{t+1}) &= e^* \\ \epsilon_{t+2}(h_0, s_0, s^{t+1}, .) &= \epsilon^*(.) \end{aligned} \tag{A26}$$

for all (h_0, s_0, s^{t+1}).

To prove that (A26) holds, we repeat, mutatis mutandis, the contradiction argument made in step 3. Specifically, suppose that there is an optimal allocation with $e_{t+1}(\bar{h}_0, \bar{s}_0, \bar{s}^{t+1}) =$

$\bar{e} < e^*$ for some $(\bar{h}_0, \bar{s}_0, \bar{s}^{t+1})$. Consider an alternative allocation that is identical to the original allocation except that (X_{h0} is decreased in order to increase c_0 for all (h_0, s_0). Further, the shortfall in production in the subsequent periods is made up by a decrease in $X_{h1}, \ldots, X_{h,t}$ until production is increased to the level of the original allocation by increasing $e_{t+1}(\bar{h}_0, \bar{s}_0, \bar{s}^{t+1})$, where $\epsilon_{t+2}(\bar{h}_0, \bar{s}_0, \bar{s}^{t+1}, .)$ is adjusted to ensure incentive compatibility. By construction, the new allocation is resource feasible. We will argue that the new allocation is also incentive compatible and does not change the continuation lifetime utility at $(\bar{h}_0, \bar{s}_0, \bar{s}^{t+1})$. Since it increases c_0 it increase social welfare and therefore contradicts the claim that the original allocation with $e_{t+1}(\bar{h}_0, \bar{s}_0, \bar{s}^{t+1}) = \bar{e} < e^*$ is socially optimal.

The argument that the increase in $e_{t+1}(\bar{h}_0, \bar{s}_0, \bar{s}^{t+1})$ does not change the continuation utility for $(\bar{h}_0, \bar{s}_0, \bar{s}^{t+1})$ is, mutatis mutandis, the same as the argument made in step 3 if the new $\epsilon'_{t+2}(\bar{h}_0, \bar{s}_0, \bar{s}^{t+1}, .)$ is chosen as the solution to

$$\begin{aligned}
\max_{\epsilon_{t+2}(.)} \quad & \left[-d(e_{t+1}) + \frac{\beta}{1-\beta}\sum_{s_{t+2}} \ln(1 + r_f(\tilde{K}(e^*)) + \epsilon_{t+2}(s_{t+2}))\pi(s_{t+2}, e_{t+1}))\right] \\
s.t. \quad & \sum_{s_{t+2}} \epsilon_{t+2}(s_{t+2})\pi(s_{t+2}, e_{t+1}) = 0 \qquad \text{(A27)} \\
& d'(e_{t+1}) = \frac{\beta}{1-\beta}\sum_{s_{t+2}} \ln(1 + r_f(\tilde{K}(e^*)) + \epsilon(s_{t+2}))\frac{\partial \pi}{\partial e_{t+1}}(s_{t+2}, e_{t+1})
\end{aligned}$$

where we suppressed for notational convenience the argument $(\bar{h}_0, \bar{s}_0, \bar{s}^{t+1})$. Thus, the proof of step 4 is completed if we show that the new allocation with $\epsilon_{t+2}(\bar{h}_0, \bar{s}_0, \bar{s}^{t+1}, .)$ defined as the solution to (A25) is incentive compatible. To prove this, it suffices to show that any $\epsilon_{t+2}(\bar{h}_0, \bar{s}_0, \bar{s}^{t+1}, .)$ that is part of an optimal allocation has to solve (A25).

To prove the last claim, take an optimal allocation and suppose for some $(\bar{h}_0, \bar{s}_0, \bar{s}^{t+1}, .)$ the corresponding $\epsilon_{t+2}(\bar{h}_0, \bar{s}_0, \bar{s}^{t+1}, .)$ does not solve (A27). Consider an alternative allocation that is identical to the original allocation with two exceptions. First, $\epsilon_{t+2}(\bar{h}_0, \bar{s}_0, \bar{s}^{t+1}, .)$ is replaced by the solution to (A27), $\epsilon_{t+2}(\bar{h}_0, \bar{s}_0, \bar{s}^{t+1}, .)$. This change increase social welfare and satisfies the incentive compatibility constraint (A15) at $(\bar{h}_0, \bar{s}_0, \bar{s}^{t+1})$ and all succeeding nodes. Second, $\epsilon_{t+1}(\bar{h}_0, \bar{s}_0, \bar{s}^t, .)$ is changed so that the incentive compatibility constraint (A15) holds at $(\bar{h}_0, \bar{s}_0, \bar{s}^t, .)$ and the net effect on social welfare remains positive, which can be achieved by reducing $\epsilon_{t+1}(\bar{h}_0, \bar{s}_0, \bar{s}^t, \bar{s}_{t+1})$ until the continuation lifetime utility at $(\bar{h}_0, \bar{s}_0, \bar{s}^t)$ is back to its original level and using the freed up resources to increase $\epsilon_{t+1}(\bar{h}_0, \bar{s}_0, \bar{s}^t, s_{t+1})$ for all $s_{t+1} \neq \bar{s}_{t+1}$. The new allocation also satisfies the incentive compatibility constraint for all nodes preceding $(\bar{h}_0, \bar{s}_0, \bar{s}^t)$ since, by the induction assumption, the effort choices $e_0, \ldots, e_t$ are all equal to a constant, e^* for all nodes (h_0, s_0, s^t). This shows that any optimal

$\epsilon_{t+2}(\bar{h}_0, \bar{s}_0, \bar{s}^{t+1}, .)$ solves (A27), which complete the proof of step 4.

Step 5 The optimal effort-risk choice, $(e^*, \epsilon^*(.))$, together with the optimal capital-to-labor ratio, $\tilde{K}^* = \tilde{K}(e^*)$, are the solution to the static social planner problem (31). Further, the optimal level of initial consumption is $C_0 = (1-\beta)(1 + r_0(\tilde{K}_0, \bar{\eta}_0))\left(\tilde{K}_0 + \frac{1}{\phi}\right) H_0$ with $r_0(\tilde{K}_0, \bar{\eta}_0) = \frac{\tilde{K}_0}{1+\tilde{K}_0}\left(F_k(\tilde{K}_0) - \delta_k\right) + \frac{1}{1+\tilde{K}_0}\left(\phi F_h(\tilde{K}_0) + \bar{\eta}_0\right)$.

Proof. The preceding argument shows that in our search for an optimal effort-risk allocation $\{\epsilon_t, e_t\}$ we can confine attention to allocations satisfying $e_t(h_0, s_0, s^t) = e^*$ and $\epsilon_{t+1}(h_0, s_0, s^{t^1}) = \epsilon^*(s_{t+1})$ for all (h_0, s_0) and for all s^t. Straightforward algebra shows that $(e^*, \epsilon^*(.))$ and the associated $\tilde{K}^* = \tilde{K}(e^*)$ are the solution to the static social planner problem (31). It remains to derive the formula for C_0.

Taking the expectations over equation (3) for the evolution of individual human capital implies that for all t

$$H_{t+1} = (1 + \bar{\eta}(e^*))H_t + \phi X_{ht} \tag{A28}$$

where $\bar{\eta}(e^*) = \sum_s \eta(s)\pi(s, e^*)$. Using (A12) in conjunction with the aggregate accumulation equation for physical capital (10), the aggregate resource constraint (11) and the aggregate production function (2) yields for all t:

$$C_t + K_{t+1} + \frac{1}{\phi}H_{t+1} = (1-\delta_k)K_t + (1 + \bar{\eta}(e^*))\frac{H_t}{\phi} + F(K_t, H_t) \tag{A29}$$

Defining $W_t \doteq K_t + \frac{1}{\phi}H_t$ and using $\tilde{K}_t = \tilde{K}(e^*)$ for $t = 1, \ldots$, we can rewrite (A29) as

$$W_{t+1} = (1 + r_f(\tilde{K}(e^*)))W_t - C_t \tag{A30}$$

for all $t = 1, \ldots$ and

$$W_1 = (1 + r_0(\tilde{K}_0, \bar{\eta}_0))W_0 - C_0$$

for $t = 0$, where we used the equality-of-return condition (27). Thus, the social planner faces an aggregate resource constraint that is equivalent to an aggregate budget constraint with investment return r_f. Solving (A30) forward using $\lim_{t\to\infty} \frac{W_t}{(1+r_f)^t} = 0$ and $r_0(\tilde{K}_0, \bar{\eta}_0) \leq r_f(\tilde{K}(e^*))$ shows that the present-value budget constraint

$$\sum_{t=0}^{\infty} \frac{C_t}{(1 + r_f(\tilde{K}(e^*)))^t} = (1 + r_0(\tilde{K}_0, \bar{\eta}_0))W_0 \tag{A31}$$

has to hold. Using the characterization (A12) for aggregate consumption we find

$$\frac{C_0}{1-\beta} \quad = \quad (1 + r_0(\tilde{K}_0, \bar{\eta}_0))W_0 \ . \tag{A32}$$

This proves the claim about C_0 and completes the proof of proposition 3.

Proof of Proposition 4.

Take a $(e^*, \epsilon^*, \tilde{K}^*)$ that solves (31). Fr given $(e^*, \epsilon^*, \tilde{K}^*)$, there is a unique (τ^*, tr^*) solving (36). To see this, note that the first equation in (36) defines the transfer payments $tr^*(s)$ for all s. The second and third equation then determine the values of τ_k^* and τ_h^*. Using proposition 1, we find that $(e^*, \epsilon^*, \tilde{K}^*)$ is an equilibrium allocation. This proves proposition 4.

Proof of Proposition 5.

Denote by $(e^*, \epsilon^*, \tilde{K}^*)$ the optimal allocation before the change in η. This $(e^*, \epsilon^*, \tilde{K}^*)$ is still in the choice set, **A**, of the social planner problem (14) after the increase in η. This proves the proposition since welfare cannot decrease in a one-agent decision problem when the choice set is changed so that the old maximizer remains in the choice set.

Job Displacement Risk

Job displacement risk is defined by the likelihood of job displacement (the job displacement rate) and the consequences of job displacement. Using the DWS data, Farber (1997) reports an average annual job displacement rate of .0384 for workers of age 35-44, which is in accordance with the job displacement rates reported by Stephens (1997) using the PSID data. Note that the job displacement rates reported in the DWS and the PSID are likely to be under-estimates of the true job displacement probabilities because of recall bias (Topel, 1991). Guided by this evidence, we use an annual job displacement rate of 4 percent as target value for $\eta_l(e)$. Note that standard measures of total rates of job separation are much larger than the job displacement rates used here. For example, Shimer (2005) estimates a monthly job separation rate for the U.S. of .034. This translates into an annual job separation rate of .49, which is an order of magnitude larger than the average job displacement rate of .04 we use in this paper.

We next turn to the consequences of job displacement, which in the model economy amount to the human capital loss, $\eta_l h$. To relate this human capital loss to the empirical literature on job displacement, note first that earnings (labor income) before taxes and transfers in the model economy are given by $y_t = \phi r_h h_t$. Thus, using the equilibrium evolution for human capital in proposition 1 we find the following expression for earnings growth of individual workers:

$$\frac{y_{t+1}}{y_t} \;=\; \beta\left(1 + \theta(1-\tau_k)r_f \;+\; (1-\theta)\left((1-\tau_h + tr(s_t))\phi\, r_h + \eta(s_t)\right)\right) \tag{A33}$$

Equation (A33) says that labor income changes associated with the displacement event are unpredictable since $\eta_t = \eta(s_t)$ and $tr_t = tr(s_t)$ define sequences of i.i.d. random variables. In other words, log-earnings follow a random walk with drift

$$\ln y_{t+1} \;=\; b \;+\; \ln y_t \;+\; \tilde{\eta}_t \tag{A34}$$

with constant drift $b = \theta(1-\tau_k)r_f + (1-\theta)(1-\tau_h)\phi\, r_h$ and an innovation term $\tilde{\eta}_t = (1-\theta)\left(tr(s_t))\phi\, r_h + \eta(s_t)\right)$, where we used the approximation $\ln(1+x) \approx x$. Thus, earnings shocks, η, are permanent and the realization $\tilde{\eta}_l = (1-\theta)\eta_l$ therefore captures the long-term decline in (pre-transfer) earnings experienced by displaced workers.

There are many studies of the long-term consequences of job displacement for U.S. workers. One of the most thorough studies on the consequences of job displacement is Jacobsen, LaLonde, and Sullivan (1993), who use longitudinal data on the earnings of high-tenure workers (workers with at least six years of tenure) in Pennsylvania from 1974 to 1986 to estimate the earnings losses of displaced workers. In their restricted sample, they confine attention to workers that are separated from distressed firms (employment contraction of at least 30%). For these workers, they find an initial drop of earnings of around 50% of pre-displacement earnings. Moreover, even though earnings recover for the first three years after displacement, this recovery is far from perfect. Indeed, six years after displacement earnings are still 25% below pre-displacement earnings. Long-term earnings losses of around 25% for high-tenure workers are in line with the estimates obtained by Topel (1990).[17]

The preceding discussion dealt with high-tenure workers, but low-tenure workers also experience substantial long-term earnings losses after job displacement. For example, Lorie and Farlie (2003) study the earnings losses of young adult workers based on the National Longitudinal Survey of Youth (NLSY), and find that the long-term earnings losses of male

[17]Long-term earnings losses experienced by displaced workers have two components: the direct decline in earnings and the forgone increase in earnings experienced by non-displaced workers. The numbers cited here refer to the total long-term earnings losses.

young adults are around 10% (for female young adults they find significantly larger losses). Couch and Placzek (2010), Davis and von Wachter (2011), Farber (2005) and Ruhm (1991) provide further evidence that long-term earnings losses for all types of displaced workers are substantial. Ruhm (1991) uses earnings data from the Panel Study of Income Dynamics (PSID) for the years 1969-1982 and finds that for a sample of displaced workers of all tenure levels (low and high tenure) the earnings losses are between 11% and 15% four years after job separation. Analyzing data drawn from the Displaced Workers Survey (DWS) between 1984-2004, Farber (2005) estimates for a sample of displaced workers of all tenure levels earnings losses of around 13%. Finally, recent work by Couch and Placzek (2010) using longitudional data for Conneticut workers and Davis and von Wachter (2011) drawing on longitudinal Social Security records for U.S. workers from 1974 to 2008 find long-term earnings losses of around $13 - 15\%$ averaged over all age- and tenure-groups.

In sum, the empirical literature suggests that job displacement leads to long-term earnings losses of up to 25% for high-tenure workers and around 10% for low tenure workers. Empirical studies that do not distinguish between low- and high-tenure workers tend to find long-term earnings losses of around $13 - 15\%$ of pre-displacement earnings. Couch and Placzek (2010) provides a recent survey of the literature reaching the same conclusion. None of the studies takes into account that job displacement often leads to a reduction in health and pension benefits. For example, if we assume that these benefits are around 15% of reported earnings, then a benefit loss of 2% of pre-displacement earnings should be added to the estimated long-term earnings losses. Guided be these considerations, for the baseline economy we choose a value of 15% for the earnings losses of displaced workers: $\tilde{\eta}_l = 0.15$.