Da Summation Sign

1 Introduction

In economics and econometrics we often work with sums. Consider the sum of the first 10 integers

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

When the number of terms gets larger this is quite cumbersome to write down. The summation sign provides a short hand for the above sum:

$$\sum_{i=1}^{10} i = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

We read this as "sum of i for i from 1 to 10". We see that we are using i as an index which runs from 1 to 10. This index is quite flexible as the following examples show

$$\begin{aligned}
\sum_{i=1}^{N} i &= 1 + 2 + 3 + \cdots + N \\
\sum_{i=-N}^{N} i &= -N - (N-1) - \cdots - 0 + 1 + 2 + \cdots N \\
\sum_{i=1}^{\infty} i &= 1 + 2 + 3 + \cdots + \cdots \\
\sum_{i=1}^{N} i^2 &= 1 + 4 + 9 + 16 + \cdots + N^2 \\
\sum_{i=0}^{N} (-1)^i (i+1)^2 &= 1 - 4 + 9 - 16 \cdots + (-1)^N (N+1)^2
\end{aligned}$$

We see that the operator is very flexible and can accommodate a variety of sums. We can also use the index i as an index for (random) variables. Consider rolling a die N times. We want an expression for the sample average (called $\bar{X}$):

$$\begin{aligned}
\bar{X} &= \frac{x_1 + x_2 + \cdots + x_N}{N} \\
&= \frac{1}{N} x_1 + \frac{1}{N} x_2 + \cdots + \frac{1}{N} x_N \\
&= \frac{1}{N} (x_1 + x_2 + \cdots + x_N)
\end{aligned}$$

We can use the summation sign to simplify these expressions and write

$$\begin{aligned}
\bar{X} &= \frac{x_1 + x_2 + \cdots + x_N}{N} = \frac{\sum_{i=1}^{N} x_i}{N} \\
&= \frac{1}{N}x_1 + \frac{1}{N}x_2 + \cdots + \frac{1}{N}x_N = \sum_{i=1}^{N} \frac{1}{N} x_i \\
&= \frac{1}{N}(x_1 + x_2 + \cdots + x_N) = \frac{1}{N}\sum_{i=1}^{N} x_i
\end{aligned}$$

It is the last expression that is most commonly used. The multiple ways of writing the sample mean brings us to our first rule considering the summation sign

Theorem 1.1. *Constants can be "pulled" in front of the summation sign. Let A be a constant. Then*

$$\sum_{i=1}^{N} A x_i = A \cdot \sum_{i=1}^{N} x_i$$

Proof:

$$\begin{aligned}
\sum_{i=1}^{N} A x_i &= A x_1 + A x_2 + \cdots + A x_N \\
&= A(x_1 + x_2 + \cdots + x_N) \\
&= A \cdot \sum_{i=1}^{N} x_i
\end{aligned}$$

For the sample average we see that $A = \frac{1}{N}$ so that we can "pull" $\frac{1}{N}$ out of the summation sign. It is also worth noting that

$$\sum_{i=1}^{N} A_i x_i = A_1 x_1 + A_2 x_2 + \cdots + A_N x_N \neq A_i \sum_{i=1}^{N} x_i$$

The reason for this is that A_i potentially differs for every x_i so that we cannot factor it out as in the second line of the proof.

What follows are some rules (called theorems here) that we will use throughout the course. Read them, go through the proofs and commit them to memory.(That's an order)

Theorem 1.2. *Let A be a constant. Then*

$$\sum_{i=1}^{N} A = N \cdot A$$

Proof:

$$\begin{aligned}\sum_{i=1}^{N} A &= \underbrace{A + A + A + ... + A}_{N\ times} \\ &= N \cdot A\end{aligned}$$

Theorem 1.3. *The sum of N numbers equals N times their average.*

$$\sum_{i=1}^{N} x_i = N\bar{X}$$

Proof:

$$\begin{aligned}\sum_{i=1}^{N} x_i &= \frac{N}{N}\sum_{i=1}^{N} x_i \\ &= N\frac{1}{N}\sum_{i=1}^{N} x_i \\ &= N \cdot \bar{X}\end{aligned}$$

Theorem 1.4. *The summation sign can be "moved through" a sum.*

$$\sum_{i=1}^{N}(x_i + y_i) = \sum_{i=1}^{N} x_i + \sum_{i=1}^{N} y_i$$

Proof:

$$\begin{aligned}\sum_{i=1}^{N}(x_i + y_i) &= (x_1 + y_1) + (x_2 + y_2) + \cdots + (x_N + y_N) \\ &= (x_1 + x_2 + \cdots + x_N) + (y_1 + y_2 + \cdots + y_N) \\ &= \sum_{i=1}^{N} x_i + \sum_{i=1}^{N} y_i\end{aligned}$$

Theorem 1.5. *Deviations from the mean sum to zero.*

$$\sum_{i=1}^{N}(x_i - \bar{X}) = 0$$

Proof:

$$\begin{aligned}\sum_{i=1}^{N}(x_i - \bar{X}) &= \sum_{i=1}^{N} x_i - \sum_{i=1}^{N} \bar{X} && \textit{by Theorem (1.4)} \\ &= \sum_{i=1}^{N} x_i - N \cdot \bar{X} && \textit{by Theorem (1.2)} \\ &= N \cdot \bar{X} - N \cdot \bar{X} && \textit{by Theorem (1.3)} \\ &= 0\end{aligned}$$

Theorem 1.6.

$$\sum_{i=1}^{N}(x_i - \bar{X})^2 = \sum_{i=1}^{N}(x_i - \bar{X})x_i = \sum_{i=1}^{N} x_i^2 - N \cdot \bar{X}^2$$

Proof:

$$\begin{aligned}
\sum_{i=1}^{N}(x_i - \bar{X})^2 &= \sum_{i=1}^{N}(x_i - \bar{X})(x_i - \bar{X}) \\
&= \sum_{i=1}^{N}(x_i - \bar{X})x_i - \sum_{i=1}^{N}(x_i - \bar{X})\bar{X} \\
&= \sum_{i=1}^{N}(x_i - \bar{X})x_i - \bar{X} \underbrace{\sum_{i=1}^{N}(x_i - \bar{X})}_{=0 \textit{ by Theorem (1.5)}} \\
&= \sum_{i=1}^{N}(x_i - \bar{X})x_i \qquad \textit{first equality} \\
&= \sum_{i=1}^{N}(x_i^2 - \bar{X}x_i) \\
&= \sum_{i=1}^{N} x_i^2 - \sum_{i=1}^{N} \bar{X}x_i \\
&= \sum_{i=1}^{N} x_i^2 - \bar{X}\sum_{i=1}^{N} x_i \qquad \textit{by Theorem (1.1)} \\
&= \sum_{i=1}^{N} x_i^2 - \bar{X}N\bar{X} \qquad \textit{by Theorem (1.3)} \\
&= \sum_{i=1}^{N} x_i^2 - N\bar{X}^2 \qquad \textit{second equality}
\end{aligned}$$

Alternative proof of second equality:

$$\begin{aligned}
\sum_{i=1}^{N}(x_i-\bar{X})^2 &= \sum_{i=1}^{N}(x_i^2-2x_i\bar{X}+\bar{X}^2) \\
&= \sum_{i=1}^{N}x_i^2-\sum_{i=1}^{N}2x_i\bar{X}+\sum_{i=1}^{N}\bar{X}^2 \qquad \textit{by Theorem (1.4)} \\
&= \sum_{i=1}^{N}x_i^2-2\bar{X}\sum_{i=1}^{N}x_i+N\cdot\bar{X}^2 \qquad \textit{by Theorems (1.2) and (1.1)} \\
&= \sum_{i=1}^{N}x_i^2-2\bar{X}N\cdot\bar{X}+N\cdot\bar{X}^2 \qquad \textit{by Theorem (1.3)} \\
&= \sum_{i=1}^{N}x_i^2-2N\bar{X}^2+N\cdot\bar{X}^2 \\
&= \sum_{i=1}^{N}x_i^2-N\bar{X}^2
\end{aligned}$$

Theorem 1.7.

$$\sum_{i=1}^{N}(x_i-\bar{X})(y_i-\bar{Y})=\sum_{i=1}^{N}(x_i-\bar{X})y_i=\sum_{i=1}^{N}x_iy_i-N\bar{X}\bar{Y}$$

The proof of the first equality follows from Theorem (1.6)
Proof of second equality:

$$\begin{aligned}
\sum_{i=1}^{N}(x_i-\bar{X})(y_i-\bar{Y}) &= \sum_{i=1}^{N}(x_iy_i-x_i\bar{Y}-y_i\bar{X}+\bar{Y}\bar{X}) \\
&= \sum_{i=1}^{N}x_iy_i-\sum_{i=1}^{N}x_i\bar{Y}-\sum_{i=1}^{N}y_i\bar{X}+\sum_{i=1}^{N}\bar{Y}\bar{X} \\
&= \sum_{i=1}^{N}x_iy_i-\bar{Y}\sum_{i=1}^{N}x_i-\bar{X}\sum_{i=1}^{N}y_i+N\cdot\bar{Y}\bar{X} \\
&= \sum_{i=1}^{N}x_iy_i-N\bar{Y}\bar{X}-N\bar{Y}\bar{X}\ +N\cdot\bar{Y}\bar{X} \\
&= \sum_{i=1}^{N}x_iy_i-N\bar{Y}\bar{X}
\end{aligned}$$

Alternatively, we can start from the first equality

$$\begin{aligned}
\sum_{i=1}^{N}(x_i - \bar{X})y_i &= \sum_{i=1}^{N}(x_i y_i - \bar{X}y_i) \\
&= \sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} \bar{X}y_i \\
&= \sum_{i=1}^{N} x_i y_i - \bar{X}\sum_{i=1}^{N} y_i \qquad \textit{by Theorem (1.1)} \\
&= \sum_{i=1}^{N} x_i y_i - \bar{X}N\bar{Y} \qquad \textit{by Theorem (1.3)} \\
&= \sum_{i=1}^{N} x_i y_i - N\bar{X}\bar{Y}
\end{aligned}$$

Theorem 1.8.

$$\sum_{i=1}^{N}\sum_{j=1}^{M} x_i y_j = \sum_{j=1}^{M} y_j \sum_{i=1}^{N} x_i = \sum_{i=1}^{N} x_i \sum_{j=1}^{M} y_j$$

Proof of first equality:

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{j=1}^{M} x_i y_j &= (x_1y_1 + x_1y_2 + \cdots + x_1y_M) \\
&+ (x_2y_1 + x_2y_2 + \cdots + x_2y_M) \\
&+ \cdots \\
&+ (x_Ny_1 + x_Ny_2 + \cdots + x_Ny_M) \\
&= x_1(y_1 + y_2 + \cdots + y_M) \\
&+ x_2(y_1 + y_2 + \cdots + y_M) \\
&+ \cdots \\
&+ x_N(y_1 + y_2 + \cdots + y_M) \\
&= x_1\sum_{j=1}^{M} y_j + x_2\sum_{j=1}^{M} y_j + \cdots + x_N\sum_{j=1}^{M} y_j \\
&= \sum_{j=1}^{M} y_j(x_1 + x_2 + \cdots + x_N) \\
&= \sum_{j=1}^{M} y_j \sum_{i=1}^{N} x_i
\end{aligned}$$

Proof of second equality:

$$
\begin{aligned}
\sum_{i=1}^{N}\sum_{j=1}^{M} x_i y_j &= (x_1y_1 + x_1y_2 + \cdots + x_1y_M) \\
&+ (x_2y_1 + x_2y_2 + \cdots + x_2y_M) \\
&+ \cdots \\
&+ (x_Ny_1 + x_Ny_2 + \cdots + x_Ny_M) \\
&= (y_1x_1 + y_1x_2 + \cdots + y_1x_N) \\
&+ (y_2x_1 + y_2x_2 + \cdots + y_2x_N) \\
&+ \cdots \\
&+ (y_Mx_1 + y_Mx_2 + \cdots + y_Mx_N) \\
&= y_1(x_1 + x_2 + \cdots + x_N) \\
&+ y_2(x_1 + x_2 + \cdots + x_N) \\
&+ \cdots \\
&+ y_M(x_1 + x_2 + \cdots + x_N) \\
&= y_1\sum_{i=1}^{N} x_i + y_2\sum_{i=1}^{N} x_i + \cdots + y_M\sum_{i=1}^{N} x_i \\
&= \sum_{i=1}^{N} x_i(y_1 + y_2 + \cdots + y_M) \\
&= \sum_{i=1}^{N} x_i \sum_{j=1}^{M} y_j
\end{aligned}
$$