Wealth Dynamics and Asset Prices with Recursive Preferences and Heterogeneous Beliefs

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Abstract

The market selection hypothesis by Alchian (1950) and Friedman (1953) implies that traders with incorrect beliefs will drop out of the market in the long-run and thus do not have any impact on prices in financial markets. In this paper I study the market selection hypothesis in an economy with heterogeneous beliefs and recursive preferences. To model heterogeneous beliefs we follow Kurz (1994) and restrict the beliefs to the subset of rational beliefs. Furthermore, households ability to borrow is limited by a collateral constraint and hence markets are incomplete. Numerical results indicate that agents with non-rational expectations do survive. In particular, more disagreement can lead to a less volatile wealth dynamics which also affects asset prices. In particular, the relationship between the tightness of the margin requirement and volatility is not monotonic.

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1 Introduction

The empirical literature on financial markets reports several paradoxes which cannot be explained by standard models such as the excess volatility puzzle (Shiller (1981)) or the equity premium puzzle (Mehra and Prescott (1985)). The literature has identified several ways to explain these paradoxes, these possibilities include incomplete markets (e.g., Telmer (1993)), non-standard preferences (for example habit formation, see e.g. Constantinides (1990) or Campbell and Cochrane (1999)) or heterogeneous beliefs (e.g., Detemple and Murthy (1994) or Basak (2005)). However, even though heterogeneous expectations have some success in explaining these paradoxes, the market selection hypothesis as formulated by Alchian (1950) and Friedman (1953) cast some doubts whether heterogeneous expectations are a suitable explanation for some of these paradoxes. They argue that in competitive markets agents with incorrect expectations make investment mistakes and lose their wealth. As their wealth depletes they drop out of the market and only investors with correct beliefs survive. Hence, investors with wrong expectations should not have any impact on prices in the long-run.

A first formal study on the market selection hypothesis was carried out by De Long et al. (1990) who showed in a partial equilibrium setting that irrational noise traders may not only survive in the long-run but can also dominate the market. Blume and Easley (2006) argue that with complete markets and time-and state separable preferences households with incorrect beliefs may not survive in the long run and thus seem to confirm the market selection hypothesis[1]. A different picture emerges when we move from an economy with complete markets to economies without a full set of Arrow-Debreu securities. In numerical examples Cao (2013) and Cogley, Sargent, and Tsyrennikov (2014) show that less informed agents may not only survive in the long-run but may also dominate the market[2]. These sharp differences in predictions on the sur-

[1]Other papers studying the survival of agents with complete markets and separable preferences include Blume and Easley (1992), Sandroni (2000), Kogan et al. (2009), Fedyk, Heyerdahl-Larsen, and Walden (2013), Cvitanic and Malamud (2010) and Cvitanić and Malamud (2011)

[2]Other papers studying economies with incomplete markets and CRRA-preferences, such as Coury and Sciubba (2012), come to similar conclusions.

vival of the less informed agent arises from the fact that due to the lack of Arrow-Debreu securities better informed agents are not able to exploit the less informed agents. These differences in results imply that market structure, i.e. the set of tradable assets and attainable portfolios, is an important factor in determining survival of agents.

Easley and Yang (2015) and Borovicka (2016) extend the analysis to economies with recursive preferences and complete markets. Recursive preferences allow the separation between risk aversion and elasticity of intertemporal substitution and have become a major workhorse in asset pricing theory. Borovicka (2016) decomposes the determinants of the wealth dynamics into three components: (i) speculative volatility, (ii) risk premium and (iii) precautionary savings. Because of the lack of full consumption insurance, precautionary savings are higher and interest rate is lower. Thus, moving from complete market to incomplete markets should affect the distribution of financia wealth in the long-run and the equilibrium properties of prices.

In this paper we study an infinite horizon exchange economy with recursive preferences, heterogeneous beliefs and incomplete markets to assess how market incompleteness affects the long-run distribution of financial wealth and how it affects prices in financial markets. There are two assets in the economy, one asset is a dividend paying real asset modelled as a Lucas tree as in Lucas (1978). The other asset is a bond and household can borrow from each other by selling bonds, however selling of the bond has to be collateralized by the real asset resulting in incomplete financial markets. Additionally the economy is inhabitate by infinitely lived agents who disagree about the probability of aggregate endowment shocks. Furthermore, they are endowed with recursive preferences that allow us to disentangle the effects of risk aversion and elasticity of intertemporal substitution on wealth dynamics and prices in the economy.

To model heterogeneous beliefs, we follow Kurz (1994) and restrict the set of possible beliefs to the subset of rational beliefs[3]. Compared to Rational Ex-

[3]There is also a more technical difference betwen my paper and Borovicka (2016). Under the rational beliefs principle non-convergence of beliefs stems from the fact that households assume non-stationarity of the underlying stochastic process, while Borovicka (2016) assumes

pectations, Rational Beliefs has weaker requirements on the knowledge of the agents. Under Rational Expectations agents know the true underlying data-generating process and form their expectations accordingly, while under Rational Beliefs agents do not know the true underlying data-generating process but use the empirical distribution to form their beliefs. If the economy is not stationary (or at least agents belief that it is not stationary), then beliefs of the agents may not converge and thus agents do hold different beliefs about the transition probabilities of exogenous variables.

Simulation results indicate that not only the elasticity of intertemporal affects the distribution of wealth in the economy but also the market structure. In particular, if the market for bonds is shut down agents cannot borrow and thus the elasticity of intertemporal has no effects on the distribution of financial wealth. If agents are instead allowed to borrow the financial wealth is affected by the precautionary savings motive, i.e. agents who don't believe that the empirical distribution is the true distribution and a low elasticity of intertemporal substitution lose more wealth in the long-run than agents with a high elasticity of intertemporal substitution.

Furthermore, simulation results show that there is a non-monotonic relationship between the tightness of the margin requirements and volatility of asset prices. This non-monotonicity is due to a trade-off between speculative trade on financial markets and the probability that a fire-sale of assets happens and a looser margin requirement implies that there is more volatility due to speculative trade but the probability of a fire-sale decreases which decreases volatility. Therefore, the volatility of asset prices is highest for mid-range values of margin requirements.

Literature

Harrison and Kreps (1978) initiated the literature on asset pricing models with heterogeneous beliefs. In general the literature on asset pricing with heterogeneous beliefs falls into two distinct camps. In one strand of literature diversity

that beliefs are not equivalent, i.e. agents disagree on the null-sets of the underlying probability space, to ensure non-convergence of beliefs.

of beliefs stems from diversity of private information (see e.g. Kyle (1985), Wang (1993), Wang (1994)), i.e. agents are still fully rational but due differences in information they form different expectations about future payoffs.

The other strand of literature assumes that agents do not form their expectations in a rational way. In particular, they assume that agents are subject to various behavioral biases (see e.g. Shefrin (2010) for a recent survey). Often these biases are incorporated in a model in a rather ad-hoc way. Furthermore, the literature has identified a plethora of behavioral biases and often the results depend on the particular bias chosen.

A middle ground between rational expectations and behavioral models is the theory of rational beliefs by Kurz (1994). He proposed a framework in which agents do use all available information but as agents think that the underlying economy is not stationary they do come to different conclusions despite having access to the same information. However, beliefs of the agents are not arbitrary but have to satisfy a rationality condition. In particular, the beliefs of the agents cannot be rejected by observable data. The theory of rational beliefs has been used to explain various asset pricing puzzles such as excess volatility Kurz and Motolese (2001), the equity premium puzzle Kurz and Beltratti (1996) or time-varying risk premia Kurz and Motolese (2011). These models typically stick to time-and state-separable preferences, whereas our paper extends this literature to the general class of recursive preferences which include time-and-state separable preferences as a special case.

This paper is also related to the large and growing literature on financial frictions. Early models include Kiyotaki and Moore (1997) and Bernanke, Gertler, and Gilchrist (1999) who show that borrowing against collateral can amplify the volatility of endogeneous variables. Aiyagari and Gertler (1999) apply the idea to asset pricing models and show that prices in economies with borrowing constraints can deviate substantially from frictionless models. In a series of papers Geanakoplos (2010), Fostel and Geanakoplos (2008) study the implications of collateral constraints for asset prices. To derive analytical solutions they have to make rather restrictive assumptions on preferences, endowments and beliefs. Cao (2013) extends this idea to economies with heterogeneous beliefs but with time-and state separable preferences while Brumm et al. (2015) use recursive preferences but homogeneous expectations. In our model we have heterogeneous beliefs and recursive preferences.

The rest of the paper is structured as follows: In Section 2 we outline the model and Section **??** discusses qualitative properties of rational belief equilibria. In Section 3 we discuss numerical results regarding survival and asset prices. Section 4 concludes the paper.

2 The Model

2.1 The Optimization Problem of the Agents

Consider an endowment economy with a single consumption good in infinite horizon. Time runs from $t = 0$ to ∞ and there are H types of consumers:

$$h \in \mathcal{H} = \{1, 2, ..., H\}$$

in the economy. These consumers might differ in many dimensions including their preferences and their endowment e_t^h of the consumption good. The consumers might also differ in their initial endowment of a real asset that pays off real dividends. However, in this paper we focus on the heterogeneity of beliefs. There are S possible exogeneous states:

$$s \in \mathcal{S} = \{1, 2, ..., S\}.$$

The state captures both aggregate uncertainty (e.g., dividends) and idiosyncratic shocks. We assume that the shocks follow a stable and ergodic markov-process with transition probabilities $\pi(s, s')$.

Real asset: There is one real asset in the economy that pays off state-dependent real dividends $\tilde{d}(s_t)$. Agents can purchase θ_t^h units of the real asset, which can also be used as collateral for borrowing. The ex-dividend price of the asset in history s^t is denoted by $\tilde{q}_t$. Consumers are also not allowed to short sell the real asset. Furthermore, the total supply of the real asset is 1.

Bond: In addition to purchasing real assets, consumers can also borrow subject to a collateral constraint. The agents borrow by selling $\tilde{b}_t^h$ units of a one-period bond which pays one unit of the consumption good in the next period at price p_t and use their holdings of the real asset as collateral. In particular, we consider a collateral constraint of the following form:

$$\tilde{b}_t^h + (1-m)\theta_t^h \min(\tilde{q}_{t+1} + \tilde{d}_{t+1}) \geq 0. \tag{1}$$

Here, m can be interpreted as the margin requirement.

Consumers: We are now turning our attention to the consumers. First, we make the following assumptions:

Assumption 1. *1. Each agent believes the economy is Markovian.*

2. Each agent believes that no single agent can affect the equilibrium.

In each period t, each consumer is endowed with some endowment $\tilde{e}_t^h$ units of the consumption good. The aggregate endowment in the economy is $\bar{e}_t = \sum_{h\in\mathcal{H}} \tilde{e}_t^h + \tilde{d}_t$ and the growth rate is denoted by $g_t = \frac{\bar{e}_t}{\bar{e}_{t-1}}$. Furthermore, we assume that they have recursive preferences. With recursive preferences the temporal resolution of uncertainty matters and preferences are not separable over time. In general, recursive preferences take the following form:

$$U_t = F(c_t, CE(U_{t+1})), \tag{2}$$

where $F(\cdot,\cdot)$ is a time aggregator and $CE(\cdot)$ is the certainty equivalent. Here we focus on the form proposed as in Epstein and Zin (1989) and Weil (1990). In particular, Consumers take the sequence of prices $\{\tilde{q}_t, p_t\}$ as given and maximize the following recursive utility function:

$$U_t^h = \left((1-\beta)\left(\tilde{c}_t^h\right)^{\frac{1-\gamma^h}{\rho^h}} + \beta E_{Q_t^h}\left[\left(U_{t+1}^h\right)^{1-\gamma^h} |\mathcal{F}_t\right]^{\frac{1}{\rho^h}}\right)^{\frac{\rho^h}{1-\gamma^h}}, \tag{3}$$

with $\tilde{c}_t^h$ as the consumption in period t, β as the subjective discount factor, γ^h as the coefficient of relative risk aversion, and the Elasticity of Intertemporal Substitution $\psi^h \geq 0$. The parameter ρ^h is defined as $\rho^h := (1-\gamma^h)/(1-\frac{1}{\psi^h})$.

And Q_t^h represents the subjective (effective) beliefs of agent h subject to the information set $\mathcal{F}_t$. The maximization problem is subject to the intertemporal budget constraint:

$$\tilde{c}_t^h + \tilde{q}_t \theta_t^h + p_t \tilde{b}_t^h \leq \tilde{e}_t^h + \tilde{b}_{t-1}^h + (\tilde{q}_t + \tilde{d}_t)\theta_{t-1}^h, \tag{4}$$

the short-sale constraint on the real asset

$$\theta_t^h \geq 0, \tag{5}$$

and the margin constraint

$$\tilde{b}_t^h + (1-m)\theta_t^h \min_{s^{t+1}|s^t} (\tilde{q}_{t+1} + \tilde{d}_{t+1}) \geq 0. \tag{6}$$

Because the functional form of the preferences are homothethic and we are in an economy with stochastic growth it is useful to rewrite the variables as a fraction of the total output $\bar{e}_t$, i.e. $q_t = \frac{\tilde{q}_t}{\bar{e}_t}$, $b_t^h = \frac{\tilde{b}_t}{\bar{e}_t}$, $e_t^h = \frac{\tilde{e}_t}{\bar{e}_t}$, $c_t^h = \frac{\tilde{c}_t}{\bar{e}_t}$. Using this notation, the optimization problem becomes[4]

$$U_t^h = \left((1-\beta) \left(c_t^h \right)^{\frac{1-\gamma^h}{\rho^h}} + \beta E_{Q_t^h} \left[\left(U_{t+1}^h g_{t+1} \right)^{1-\gamma^h} | \mathcal{F}_t \right]^{\frac{1}{\rho^h}} \right)^{\frac{\rho^h}{1-\gamma^h}}, \tag{7}$$

subject to

$$\begin{aligned} c_t^h + q_t \theta_t^h + p_t b_t^h &\leq e_t^h + \frac{b_{t-1}^h}{g_t} + (q_t + d_t)\theta_{t-1}^h, & (8)\\ \theta_t^h &> 0, & (9)\\ b_t^h + (1-m)\theta_t^h \min_{s^{t+1}|s^t} (q_{t+1} + d_{t+1}) &\geq 0. & (10) \end{aligned}$$

This description of the consumers optimization problem implies that the demand correspondence for an agent h in period t will depend on the assets he

[4]The derivation of the first order conditions are shown in appendix B

purchased in the previous period, the current state s_t as well as the beliefs Q_t^h:

$$\theta_t^h = \theta^h\left(\theta_{t-1}^h, b_{t-1}^h, p_t, q_t, s_t, Q_t^h\right), \tag{11}$$

$$b_t^h = \theta^h\left(\theta_{t-1}^h, b_{t-1}^h, p_t, q_t, s_t, Q_t^h\right). \tag{12}$$

Hence, equilibrium allocation and prices in period t depends on the distribution of assets $\left(\theta_{t-1}^h, b_{t-1}^h\right)_{h\in\mathcal{H}}$ as well as distribution of beliefs.

2.2 Equilibrium

Market Clearing Conditions: The market clearing conditions for our economy are straight forward:

1. The market for the risky asset clears

$$\sum_{h\in\mathcal{H}} \theta_t^h = 1 \quad \text{for all } t = 1, ... \tag{13}$$

2. The market for the risk-free asset clears

$$\sum_{h\in\mathcal{H}} b_t^h = 0 \quad \text{for all } t = 1, ... \tag{14}$$

3. The market for the consumption good clears

$$\sum_{h\in\mathcal{H}} c_t^h = \sum_{h\in\mathcal{H}} e_t^h + d_t \quad \text{for all } t = 1, ... \tag{15}$$

Using the market clearing conditions, we can define a Markov-Competitive equilibrium as follows:

Definition 1. *A **Markov competitive equilibrium** is defined as a sequence of probability measures $\{Q_t^1, Q_t^2, ..., Q_t^H\}_{t=1}^{\infty}$ and $(q_t, q_t^b, g_t, d_t, e_t^h, \theta_t^h, b_t^h, c_t^h)$ satisfy the first-order optimality conditions, the complementary slackness conditions for all h and t, and the equilibrium conditions for all t.*

The market clearing conditions together with the demand correspondences imply that in equilibrium prices in period t are a function of the endogeneous state variables (i.e. portfolio choices in the previous period), the current state of the economy s_t and the beliefs $(Q_t^h)_{h\in\mathcal{H}}$:

$$\begin{bmatrix} q_t \\ q_t^b \end{bmatrix} = f\left(\left(\theta_{t-1}^h, b_{t-1}^h\right)_{h\in H}, s_t, \left(Q_t^h\right)_{h\in\mathcal{H}}\right). \tag{16}$$

2.3 The Structure of Beliefs

So far, we have taken the beliefs $(Q_t^h)_{h\in\mathcal{H}}$ as given and we are now turning to the construction of rational beliefs. The key tool in the construction of rational beliefs is the introduction of generating variables which fully describe the beliefs of the agents. Then the conditional stability theorem by Kurz and Schneider (1996) states that our construction constitutes a rational belief.

First, we describe the construction of the generating variables. Let X denote the state-space of data and observable $(s_t, p_t, q_t, (\theta_t^h, b_t^h, c_t^h)_{h\in\mathcal{H}})$ for all t and X^∞ the state space for the entire sequence. The Borel σ field generated by X^∞ will be denoted as $\mathcal{B}(X^\infty)$. The true stochastic process of the economy is described by a stochastic dynamic system $(X^\infty, \mathcal{B}(X^\infty), T, \Pi)$, where T denotes the shift-transformation[5] and Π the true probability measure.

We define now a rational belief:

Definition 2. *(Rational Beliefs) A sequence of effective beliefs $\{Q_t^h\}_{t=0}^\infty$ are a rational belief if the sequence is stable and ergodic, is compatible with the data and it induces a stationary measure that is equivalent to the one induced by the empirical measure Π.*

The crucial part of this definition is that rational beliefs have to be 'compatible with the data'[6]. Intuitively the long-run distribution of the agents' beliefs has to be the same as the long-run distribution of prices and sates. Or, in other words it states that unconditional beliefs have to coincide with the empirical measure. However, rational beliefs still allows for mistakes as the definition

[5]The shift transformation T is defined as $x_{t+1} = Tx_t$. It is not assumed to be invertible, i.e. $T^{-1}x_{t+1} \neq x_t$, which implies that any future evolution is not associated with a unique past.

[6]We provide a formal definition in Appendix A

does not require the belief to the true probability. It is important to note the rational beliefs principle rules out fixed (or dogmatic) beliefs, unless they believe that the empirical distribution is the true distribution. Do note that this definition of rational beliefs does not require agents to know the equilibrium map (16), instead agents deduce the relationships between variables on the observable data.

However, the definition of rational beliefs in this sense that does not tell us how we should construct rational beliefs or how agents should learn from the available data and it merely restrict the set of possible beliefs. Furthermore, a belief on X^∞ is a rather complicated object and it may prove impossible to check stability. Hence, instead modelling the learning process we pose the problem differently: Given the dynamic system $(X^\infty, \mathcal{B}(X^\infty), T, \Pi)$ we construct a sequence of effective beliefs that are rational beliefs.

To include the beliefs of the agents we follow Kurz and Schneider (1996) expand the probability space and include the sequence of generating variables $(n_t^h)_{t=1}^\infty$. Now, agent h forms a belief Q_t^h on $\left(\left(X \times \mathcal{N}^h\right)^\infty, \mathcal{B}\left((X \times \mathcal{N})^\infty\right)\right)$, where $\mathcal{N}^h := \{0, 1\}$ denotes the state space of n_t^h, and $\mathcal{B}\left(\left(X \times \mathcal{N}^h\right)^\infty\right)$ is the Borel σ-field generated by $\left(X \times \mathcal{N}^h\right)^\infty$. Now let $n^{ht} := (n_1^h, n_2^h, ..., n_t^h)$, i.e. the history of generating variables n_t^h up to period t. Then, each finite history n^{ht} determines agent $h's$ effective belief in period t denoted by $Q_t^h(A) = Q^h(A|n^{ht})$ for $A \in \mathcal{B}\left(X^\infty\right)$, which is a probability measure on $\left(X^\infty, \mathcal{B}\left(\Sigma^\infty\right)\right)$. The analysis is simplified by the following assumption:

Assumption 2. *The marginal distribution for n_t^h with respect to Q_t^h is i.i.d. with $Q^h(n_t^h = 1) = \mu^h$.*

Assumptions 1 and 2 imply that the effective belief Q_t^h is solely determined by the generating variable n_t^h, i.e. $Q_t^h(A) = Q^h(A|n_t^h)$ for $A \in \mathcal{B}\left(\Sigma^\infty\right)$. Hence, we interpret the variable n_t^h as describing the state of belief of agent h in period t.

For example, the belief Q^h supports a regime switching model[7], then n_t^h describe the regime which agent h believes the economy is in. For instance, if

[7]Regime switching processes are popular in econometrics to model non-linearities in macroeconomic time series (e.g. Hamilton (1989))

$n_t^h = 1$ agent h may be optimistic about the economy while $n_t^h =$ corresponds to a pessimistic state of belief.

Furthermore, even though households switches between beliefs are i.i.d. the description above allows us to model a wide range of joint dynamics between beliefs and states.

In the next step we need to make sure that the conditional probabilities Q^h given $n^h \in (\mathcal{N}^\infty)$ are a rational belief. This is achieved via the Conditional Stability Theorem. However, before we state the Conditional Stability Theorem, we introduce some important notation.

Let Π_k^h denote the conditional probability of $\hat{\Pi}^h$ given a particular sequence of effective beliefs $k \in (\mathcal{N}^h)^\infty$:

$$\hat{\Pi}_k^h(\cdot) : (\mathcal{N}^h)^\infty \times \mathcal{B}(X^\infty) \mapsto [0,1]. \tag{17}$$

For each $A \in \mathfrak{B}(X^\infty)$, $\hat{\Pi}_k^h$ is a measurable function of k and for each k, $\hat{\Pi}_k^h(\cdot)$ is a probability on $(N^\infty, \mathcal{B}(X^\infty))$. For $A \in \mathcal{B}(X^\infty)$ and $B \in \mathcal{B}((\mathcal{N}^h)^\infty)$, we have

$$\hat{\Pi}^h(A \times B) = \int_{k \in B} \hat{\Pi}_k^h(A)\bar{\mu}^h(dk). \tag{18}$$

Also, as we noted above,

$$\begin{aligned} \Pi(A) &= \hat{\Pi}^h(A \times (\mathcal{N}^h)^\infty), \forall A \in \mathcal{B}(X^\infty), \\ \bar{\mu}^h(B) &= \hat{\Pi}(X^\infty \times B), \forall B \in \mathcal{B}((\mathcal{N}^h)^\infty). \end{aligned} \tag{19}$$

If $(\Omega^h, \mathfrak{B}^h, T, \hat{\Pi}^h)$ is a stable dynamical system with a stationary measure $m^{\hat{\Pi}^h}$, we define the two marginal measures of $m^{\hat{\Pi}^h}$ as follows:

$$m(A) := m^{\hat{\Pi}^h}(A \times (\mathcal{N}^h)^\infty), \forall A \in \mathcal{B}(X^\infty). \tag{21}$$

$$m_{Q^h}(B) := m^{\hat{\Pi}}(X^\infty \times B), \forall B \in \mathcal{B}((\mathcal{N}^h)^\infty). \tag{22}$$

Also, let $\hat{m}_k$ denote the stationary measure of $\hat{\Pi}_k^h$, which is a measure on $(X^\infty, \mathfrak{B}(X^\infty))$. Given the construction of the dynamical system, we have the following theorem:

Theorem 1. *(Conditional Stability Theorem, Kurz and Schneider (1996)).*

Let $(\Omega^h, \mathcal{B}^h, T, \hat{\Pi}^h)$ be a stable and ergodic dynamical system. Then,

1. *$(X^\infty, \mathcal{B}^\infty, T, \hat{\Pi}^h_k)$ is stable and ergodic for $\hat{\Pi}^h$ a.a. k.*

2. *$\hat{m}^h_k$ is independent of k, $m^h_k = m = \Pi$.*

3. *If $(X^\infty, \mathfrak{B}(X^\infty), T, \hat{\Pi}^h_k)$ is stationary, then the stationary measure of $\hat{\Pi}^h_k$ is Π. That is*

$$\hat{m}^h_k = m = \Pi.$$

So far, our discussion on constructing rational beliefs did assume that agents do not know the equilibrium map (16). However, to simplify the computational model we assume that agents do know the equilibrium map (16)[8]. This implies that once agents have chosen their portfolios next periods prices depend only on the state s_t and the distribution of belief $(Q^h_t)_{h\in\mathcal{H}}$. This simplifies the construction of a computational model as agents need to form beliefs only over the exogeneous variables, i.e. they form beliefs over $\left(s_t, (n^h_t)_{h\in\mathcal{H}}\right)$

To illustrate and clarify the construction of rational beliefs, we consider an example similar to our simulation model discussed in a later section.

Example. *Consider an economy with two exogeneous states (e.g. high dividends and low dividends) and two agents. Both agents can be either optimistic in the sense that she assigns a higher probability on higher dividends then empirically observed or pessimistic in the sense that she assigns a lower probability on high dividends than empirically observed. Now, our state-space consists of 8 states. In particular, we have the tuple $\{d_t, n^1_t, n^2_t\}$. Now, in period t agents form beliefs not only over dividends but also over the distribution of future generating variables $\{n^1_{t+1}, n^2_{t+1}\}$.*

This implies that the sequence of effective beliefs of a household Q^h_t must have the same stationary distribution as the tuple $\{d_t, n^1_t, n^2_t\}^\infty_{t=0}$. If the beliefs are represented by two transition matrices F^h_H, F^h then the beliefs of agent h are dependent on the generating variable n^h_t are as follows:

- *choose F^h_H if $n^h_t = 1$*

[8]This assumption is similar to assumptions made in the literature on bayesian learning.

- *choose F^h_L if $n^h_t = 0$*

If the tuple has a Markov transition matrix Γ and the beliefs are represented by two transition matrices F^h_H, F^h_L the rationality condition implies that

$$\mu^h F^h_H + (1 - \mu^h) F^h_L = \Gamma. \tag{23}$$

□

Now, using the generating variable, we can rewrite the portfolio choice (11)-(12) in terms of generating variables rather than beliefs Q^h_t:

$$\theta^h_t = \theta^h \left(\theta^h_{t-1}, b^h_{t-1}, p_t, q_t, s_t, n^h_t \right), \tag{24}$$

$$b^h_t = \theta^h \left(\theta^h_{t-1}, b^h_{t-1}, p_t, q_t, s_t, n^h_t \right). \tag{25}$$

From this, it follows that prices in the economy are given by

$$\begin{bmatrix} q_t \\ q^b_t \end{bmatrix} = f\left(\left(\theta^h_{t-1}, b^h_{t-1} \right)_{h \in H}, s_t, \left(Q^h_t \right)_{h \in \mathcal{H}} \right). \tag{26}$$

We define the stochastic primitives y_t as follows:

$$y_t = \left(s_t, (n^h_t)_{h \in \mathcal{H}} \right) \quad \forall t. \tag{27}$$

The state space of the stochastic primitives is now $\mathcal{Y}$. We assume that $\{y_t\}^\infty_{t=0}$ is a stable Markov process with a time homogeneous transition probability $P : \mathcal{Y} \to \mathcal{P}(\mathcal{Y})$, where $\mathcal{P}(\mathcal{Y})$ denotes the space of probability measures on $\mathcal{Y}$.

This construction of rational beliefs allows us now to define a Rational Belief Equilibrium as follows:

Definition 3. *A **Rational Belief Equilibrium** is a Markov competitive equilibrium that satisfies the rationality conditions for the sequence of beliefs $\{Q^1_t, Q^2_t, ..., Q^H_t\}^\infty_{t=1}$.*

3 Quantitative Analysis

In this section we focus on the quantitative analysis of the model. In section 3.1 we discuss how to apply the structure for rational beliefs as outlined in section 2.3 into a simulation framework and the parameterization of the model. Section 3.2 discusses the results regarding the survival of agents and section 3.3 discusses the Asset-pricing implications.

3.1 The Simulation Model

For the simulation model we assume that there are 2 agents in the economy, that is, $H = 2$. We also assume that there are two growth states, i.e. $g_t \in \{\bar{g}, \underline{g}\}$.

The empirical distribution $\{g_t\}$ follows a markov-process:

$$\Psi = \begin{bmatrix} \phi & 1-\phi \\ 1-\phi & \phi \end{bmatrix}. \tag{28}$$

The stationary transition probability matrix for the tupel (g_t, n_t^1, n_t^2) has to satisfy the following conditions:

- the empirical distribution for the process g_t is specified by transition probability matrix Ψ.
- the marginal distribution for n_t^h is i.i.d with frequency of $\{n_t^h = 1\} = \alpha^h$.

Here, we use a specification similar to Kurz and Motolese (2001) as we know that the beliefs are compatible with the stationary distribution and it can generate large fluctuations. Furthermore, this specification allows for correlation between the three variables (g_t, n_t^1, n_t^2). We assume that the 8×8 matrix Γ has the following structure:

$$\Gamma = \begin{bmatrix} \phi A & (1-\phi)A \\ (1-\phi)A & \phi A \end{bmatrix}. \tag{29}$$

A is a 4×4 matrice defined by 6 parameters (α^1, α^2, a) and $a = (a^1, a^2, a^3, a^4)$

as follows:

$$A = \begin{bmatrix} a^1 & \alpha^1 - a^1 & \alpha^2 - a^1 & 1 + a^1 - \alpha^1 - \alpha^2 \\ a^2 & \alpha^1 - a^2 & \alpha^2 - a^2 & 1 + a^2 - \alpha^1 - \alpha^2 \\ a^3 & \alpha^1 - a^3 & \alpha^2 - a^3 & 1 + a^3 - \alpha^1 - \alpha^2 \\ a^4 & \alpha^1 - a^4 & \alpha^2 - a^4 & 1 + a^4 - \alpha^1 - \alpha^2 \end{bmatrix}. \tag{30}$$

We also have to specify the transition probability matrices that represent the beliefs of the agents. As noted above, agent $h \in \{1,2\}$ in period t uses F_1^h when his generating variable is $n_t^1 = 1$ and F_2^h when his generating variable is $n_t^1 = 0$. The rationality of belief condition implies that

$$\alpha^h F_1^h + (1-\alpha^h) F_2^h = \Gamma. \tag{31}$$

Thus to fully pin down a traders' belief we only have to specify F_1^h while F_2^h can be inferred from Γ and F_1^h. The matrix F_1^h is parametrized by η^h as follows:

$$F_1^h(\eta^h) \quad = \quad \begin{bmatrix} \phi \eta^h A & (1-\eta^h \phi) A \\ (1-\phi)\eta^h A & (1-(1-\phi)\eta^h) A \end{bmatrix}. \tag{32}$$

From the above equation one can see that if $\eta^h > 1$ a trader places more weight on the growth states, i.e. he is overly optimistic that the economy grows when his beliefs are given by F_1^h. Furthermore, the larger the η^h implies a more optimistic trader. Furthermore, parameter α^h determines the frequency of optimistic beliefs, when $\alpha^h = 0.5$ then optimistic and pessimistic have the same frequency while $\alpha^h > 0.5$ implies that a trader is more often optimistic then pessimistic. This has also implications for pessimistic beliefs. In particular if $\eta^h > 1$ and $\alpha^h > 0.5$ then beliefs are more asymmetrically distributed to satisfy the rationality condition.

Following Mehra and Prescott (1985) we consider the following transition probability matrix for Ψ:

$$\Psi = \begin{bmatrix} 0.43 & 0.57 \\ 0.57 & 0.43 \end{bmatrix}, \tag{33}$$

and set $\overline{g} = 1.054$ and $\underline{g} = 0.982$. And in line with the literature we set the dividends d_t to $d_t = 0.15$.

For the beliefs of the agents we follow Kurz and Motolese (2001) and set $(a_1, a_2, a_3, a_4) = (0.5, 0.14, 0.14, 0.14)$. Furthermore, we assume that $\alpha^1 = \alpha^2 = \alpha = 0.57$. The maximum value for η is $1/0.57 \approx 1.7$ and we will examine several different cases of η. To focus on the survival aspect, we consider the case that agent 2 beliefs that the empirical distribution is the true distribution, i.e. $\eta^2 = 1$, while agent 1 does not believe that the empirical distribution is the true distribution. In particular we consider $\eta^1 \in \{1.2, 1.4, 1.6\}$.

Additionally, we also modify the process for the evolution of beliefs. In particular we look at two more parameterizations:

- Model 2: $\alpha = 0.57, \quad (a_1, a_2, a_3, a_4) = (0.26, 0.24, 0.24, 0.24)$
- Model 3: $\alpha = 0.50, \quad (a_1, a_2, a_3, a_4) = (0.26, 0.24, 0.24, 0.24)$

In model 3, the evolution of beliefs follows a process that is nearly an i.i.d. process while model 2 represents a middle case between our baseline model and the i.i.d. case. The results from these additional models are qualitatively not different from our baseline model, therefore the additional results are presented in Appendix D

Our choices for preferences follow the literature. We set the time-preference parameter to $\beta = 0.96$, the coefficient of relative risk-aversion is set to $\gamma = 1.5$ which is standard in the literature. On the other hand, for the value of the EIS there is a bigger range of estimates. Some authors estimate a rather low value for the EIS, for example Hall (1988) estimates a value much smaller than 1, while several asset pricing models (e.g., Collin-Dufresne, Johannes, and Lochstoer (2014) or Bansal and Yaron (2004)) have used a EIS greater than 1. An EIS greater than 1 is needed to capture the negative correlation between consumption volatility and the price/dividend-ratio. For the baseline model we set the elasticity of intertemporal substitution for both agents to $\psi = 1.5$, a value which is in line with the asset pricing literature.

Furthermore, for the numerical solution of the model it is useful to collapse the endogeneous state variables (b_t^h, θ_t^h) into one variable. We follow Kubler and Schmedders (2002) and use the financial wealth share as the endogeneous state variable in our model. The financial wealth of agent h is

$$W_t^h = \theta_{t-1}^h (q_t + d_t) + \frac{b_{t-1}^h}{g_t} \tag{34}$$

From the equilibrium conditions we deduce that the total financial wealth in the economy is $q_t + d_t$. Hence, the financial wealth share is

$$w_t^h = \frac{\theta_{t-1}^h (q_t + d_t) + \frac{b_{t-1}^h}{g_t}}{q_t + d_t} \tag{35}$$

Because of the short-sale constraint on stocks and the collateral requirement it is easy to see that the financial wealth share is bounded between 0 and 1.

The model is solved using a policy function iteration with the details of the solution algorithm outlined in appendix C

3.2 Survival of Agents

3.2.1 Wealth Distribution

We follow the literature and define survival in terms of financial wealth. We say that a consumer becomes extinct if her financial wealth share converges to zero and survives if it doesn't converge to zero. Furthermore, a consumer dominates the market if her financial wealth converges to 1. Formally, a household h becomes extinct if

$$\lim_{t\infty} \omega_t^h = 0 \text{ almost surely,} \tag{36}$$

and survives if she doesn't become extinct. And a consumer h dominates the market if

$$\lim_{t\infty} \omega_t^h = 1 \text{ almost surely.} \tag{37}$$

All results are obtained from simulations. We simulate the economy for 100

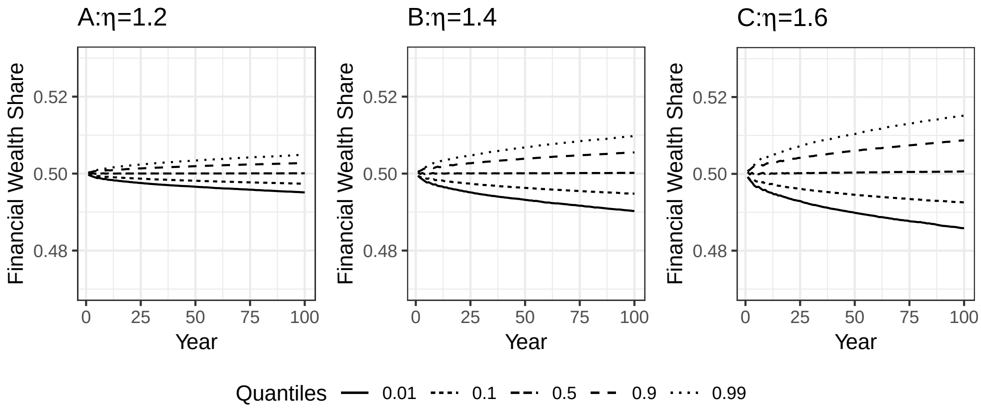

Figure 1: This graphs shows the dynamics of the wealth distribution of Agent 1 over 100 years. The quantiles of the wealth distribution are $0.01, 0.1, 0.5, 0.9, 0.99$. The initial wealthshare is $\omega^1 = 0.5$. The coefficient of relative risk-aversion is $\gamma = 1.5$ and the Elasticity of Intertemporal substitution is $\psi = 1.5$. The margin requirement is $m = 1.00$.

periods (years) and the number of simulations is N=50000.

Borovicka (2016) argues that there are three distinct channels that affect financial wealth in the long-run. In particular, these are

1. *Risk premium channel:* A higher risk aversion implies a higher risk premium from which more optimistic agents profit.

2. *Speculative Volatility channel:* If differences in beliefs become greater, agents will chose a portfolio with more volatile returns which implies that expected logarithmic returns decrease.

3. *Saving channel:* Agents with a higher Elasticity of Intertemporal Substitution save more. Furthermore, with more complete markets (in the sense that the collateral constraint becomes loser) the interest rate decreases. How agents react to the change in interest rate depends again on the EIS. In particular with a lower EIS, the substitution effect dominates the income effect and households save more with a decreasing interest rate.

We first turn our attention to the case of $m = 1.00$, i.e. agents cannot borrow.

In this case, the budget constraint for both agents reduces to

$$c_t^h + q_t \theta_t^h \leq e_t^h + (q_t + d_t)\theta_{t-1}^h. \tag{38}$$

Kehoe and Levine (2001) refer to this situation as liquidity constrained and the financial wealth share is equivalent to the position in the risky asset. As agents cannot borrow, financial frictions do not play a role and the only driver of the results are risks stemming from differences in beliefs as well as aggregate growth risk which affects marginal utility and thus affects prices today. Furthermore, there are only two channels active in this situation: the risk premium channel and the speculative volatility channel.

In Figure 1 we show the results of the simulation exercise. The three panels show the quantiles of the financial wealth distribution for different levels of disagreement, i.e. different values of η^1. One can see that for all three possible values of η^1 that after 100 years the median financial wealth share is at $\omega_{100}^1 = 0.5$, i.e. both agents survive in the long-run but neither starts to dominate the market. Furthermore, the distribution is symmetric around the median wealth share. The graphs also show that with a higher disagreement the wealth distribution becomes more dispersed.

The intutition behind this result is rather simple. If agent 1 is overly optimistic about next periods return on holdingthe asset she is keen to increase her position in the risky asset. But by investing more in the risky asset her return become more volatile. Yet, as the median wealth share is at 0.5 neither the risk premium channel nor the speculative volatilie is dominating the long-run outcome. However, with a higher η^1 agents chose more volatile portfolios and thus the long-run distribution of financial wealth becomes more dispersed.

We now move on to the case of $m = 0.50$. Now, the savings channel is also active in the model. Furthermore, aggregate growth affects also the distribution of future financial wealth. In particular, if a household holds bonds then in a recessesion (i.e. $g_{t+1} < 1$) his financial wealth increases while in a boom state (i.e. $g_t > 1$) his financial wealth decreases. Thus, the more a household has invested in bonds (or has borrowed) the more she is affected by aggregate growth risk.

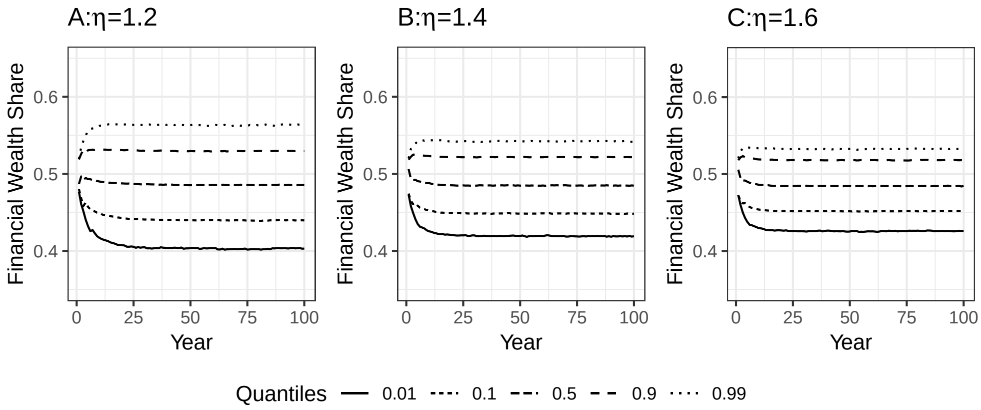

Figure 2: This graphs shows the dynamics of the wealth distribution of Agent 1 over 100 years. The quantiles of the wealth distribution are $0.01, 0.1, 0.5, 0.9, 0.99$. The initial wealthshare is $\omega^1 = 0.5$. The coefficient of relative risk-aversion is $\gamma = 1.5$ and the Elasticity of Intertemporal substitution is $\psi = 1.5$. The margin requirement is $m = 0.50$.

For the case of $m = 0.50$ we make now the following observations. First, with increasing diversity in beliefs the distribution of financial wealth becomes more narrow. Second, in contrast to the case of $m = 1.00$ the median wealth distribution drops below 0.5. Third, after some first gains or losses the quantiles of the wealth distribution are stable.

These results show that there are two different mechanisms through which growth affects the survival of agents. In the case $m = 1$, economic growth only affects marginal utility and only the choice of different asset holdings affects the results. If $m < 1$, then for any given portfolio growth affects the future endogeneous states.

3.2.2 The Role of the Elasticity of Intertemporal Substitution

In order to understand the importance of the EIS, i.e. the desire to smooth consumption, on the results we keep η^1 fixed at $\eta^1 = 1.6$ and consider now three different case of ψ. In particular, we consider the cases $\psi \in \{0.35, 1/\gamma, 1.5\}$. Figure 3 shows the dynamics of the median financial wealth over 100 years when $m = 1$ and $m = 0.50$. If there is no borrowing, the effects of changes in the EIS on the dynamics of financial wealth are negligible. This is in contrast

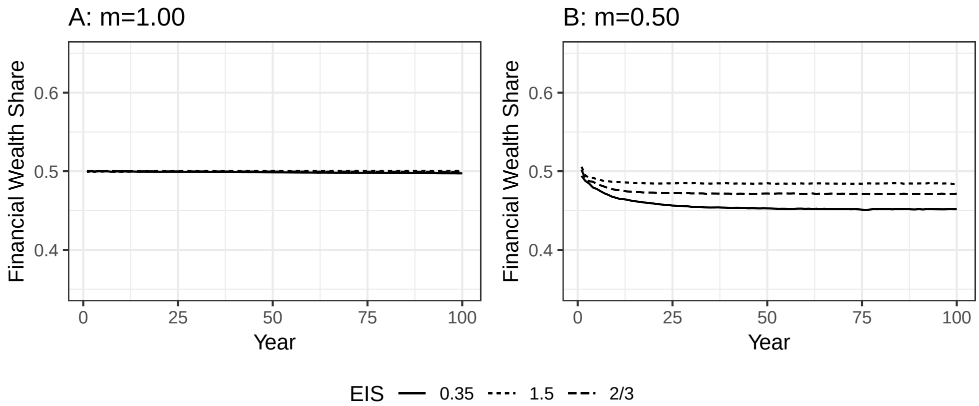

Figure 3: This figure shows the dynamics of the median financial wealth distribution over 100 years for different elasticity of intertemporal substitution. The left panel shows the dynamics for $m = 1.00$ and the right panel for $m = 0.50$. The coefficient of relative risk aversion is $\gamma = 1.5$ and $\eta^1 = 1.6$. The inital wealth share is $\omega_0^1 = 0.5$ and the the elasticity of intertemporal substitution are 0.35, 2/3 and 1.5

to the case when borrowing is allowed where the elasticity of intertemporal substitution clearly impacts the dynamics. In particular, we see that a lower EIS implies a lower median financial wealth.

This implies that the main driver of the results is the effect the EIS has on the composition on the portfolio. If there is no borrowing in the economy, the only reason to invest in the asset is the expected return. The ability to borrow changes the dynamics drastically as an additional motive to hold the risky asset comes into play. Agents now hold the risky asset in order to borrow. In order to further understand the importance of the EIS we look at the policy functions. Given the fact that agent 2 believes that the empirical distribution is the true distribution we have 4 possible states to look at.

In Figure 4 the log bond-price is shown as a function of the financial wealth shar of the agents. Obviously when agent 1's wealth share rises above 0.5 then he starts to dominate the market and her influence on prices becomes bigger. In particular we can see that when agent 1 is optimistic the interest rate is larger than in pessimistic states as in optimistic states agent 1 prefers to invest into the risky asset as he subjectively believes that there is a high rate

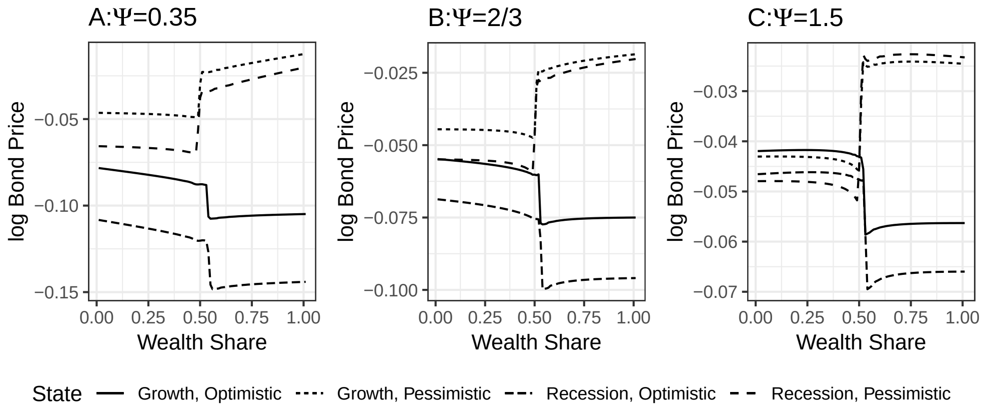

Figure 4: This figure shows the log bond price as a function of the financial wealth share for different values of Ψ. The beliefs of agent are set to $\eta^1 = 1.6$ and the collateral constraint is set to $m = 0.5$

of return. This implies that she is also more keen to borrow from agent 2, hence in equilibrium the interest rate has to rise to equilibriate the supply and demand for bonds.

This effect is exacerbated when the EIS is low as a low EIS implies a low desire to smooth consumption and hence lower precautionary savings. Thus in equilibrium interest rate decreases in optimistic states when the EIS increases. This can also be seen from Figure 5, i.e. with a higher EIS agents save more.

This in turn affects the long-run distribution of financial wealth in the economy, i.e. agents with a lower EIS will have a lower long-run financial wealth as they invest more aggresively in the risky asset and save less and thus are more hit from the malinvestment when the wrong state occurs. Consequently, if the EIS is lower than the median financial wealth is also at a lower level.

3.3 Equilibrium Asset Prices

In the previous section we have seen that the margin requirement m and the elasticity of intertemporal substitution are key determinants for the endogeneous distribution of financial wealth as they affect portfolio and savings decisions of households which in turn affects equilibrium properties of the economy. In this section we further examine the asset pricing properties of the

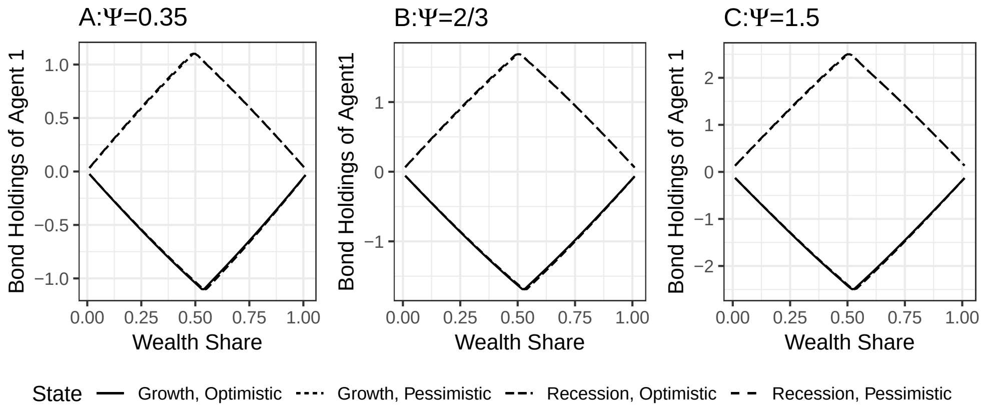

Figure 5: This figure shows the bond holdings as a function of the financial wealth share for different values of Ψ. The beliefs of agent are set to $\eta^1 = 1.6$ and the collateral constraint is set to $m = 0.5$

economy.

3.3.1 Volatility

We are now turning to the properties of equilibrium asset prices. In particular, we are looking at the unconditional volatility of asset prices and interest rates. Figure 6 shows the volatility of normalized equity prices q_t and interest rate $r_t = \frac{1}{p_t} - 1$ for three different values of η^1.

There are several factors that affect asset price volatility. First, aggregate growth risk, second, risks stemming from time-variation in beliefs. Credit cycles represent the third type of risk, i.e. if agent 1 is optimistic and believes that the economy will be growing he will borrow to buy the asset and thee price of the risky asset increases, if there is now a negative shock and the value of the risky asset declines agent 1 has to sell the asset in order to meet her obligations, thus price of the risky asset drops.

While volatility of the normalized asset prices and interest rate is always higher with more disagreement and is thus consistent with the literature, we find that relaxing the margin requirement m has non-monotonic effects on the volatility of the normalized equity prices.

As already noted, in the case $m = 1.00$ there are only two risk factors:

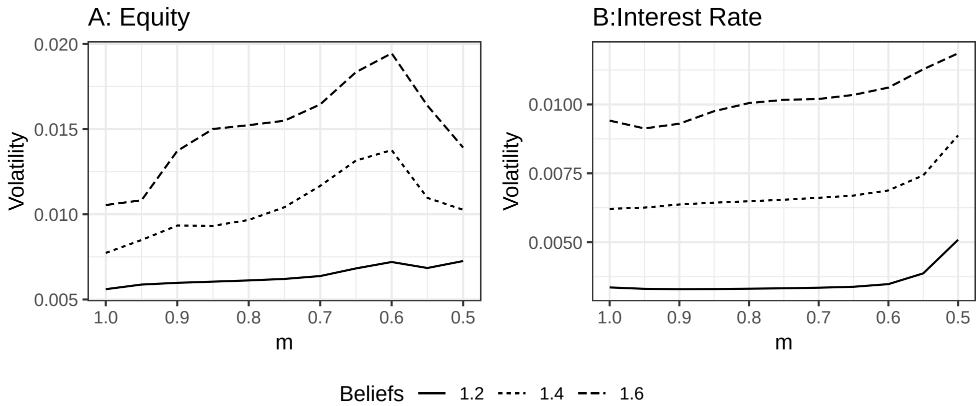

Figure 6: Volatility of equity price q_t (Panel A) and Interest Rate (Panel B) as a function of the margin requirement m. The initial wealthshare is $\omega^1 = 0.5$. The coefficient of relative risk-aversion is $\gamma = 1.5$ and the Elasticity of Intertemporal substitution is $\psi = 1.5$.

Changes in aggregate endowment as well as changes in beliefs. As no agent is driven out prices are determined by the euler-equations of both agents. Thus, larger disagreements about the aggregate growth implies a higher volatility.

If agents are allowed to borrow, i.e. $m < 1$, further risk is added. As long as the margin constraint is not binding, i.e. wealth shares do not approach the upper (1) or lower limit (0), a negative shock to economic fundamentals and in turn a negative shock to the price of the risky asset has no effects as agents are not forced to drastically change their position in bonds and stocks. If, on the other hand, the margin constraint is binding, a negative shock may trigger the fire-sale dynamics of prices. The probability whether a margin constraint is binding depends on the tightness of the margin constraint. The probability of a binding margin constraint is high when the margin constraint is strict[9] and becomes lower when the margin constraint is relaxed. Thus, there is a trade-off between the probability that a margin constraint is binding and speculative trade.

Even though the relationship between margin requirements and price volatility has been extensively studied in the literature there is no consensus whether

[9]In the case of $m = 1.00$ the margin constraint is always binding, but there are no fire-sale dynamics of asset prices.

tighter margin requirements increase or decrease volatility. For instance Hsieh and Miller (1990) and Moore (1966) find no relationship between margin requirements and volatility, while Hardouvelis (1990) and Hardouvelis and Peristiani (1992) argue that relaxing the margin requirements increases volatility.[10]

The result that relaxing the margin requirement does not necessarily lead to a strictly increasing or strictly decreasing volatility is in constrast to results under rational expectations (Brumm et al. (2015)) or dogmatic overly optimistic beliefs (Cao (2013)).

Under rational expectations agents trade on financial markets to share risks and smooth consumption. Relaxing margin requirement means that the set of attainable portfolios increases and thus improves the agents' ability to share risks and smooth consumption. With better risk-sharing, idiosyncratic risks decline and we get closer to a representative agent benchmark where only aggregate risks matter. Thus, volatility declines.

With dogmatic beliefs we have another effect. In particular agents use financial markets for speculative purposes, i.e. they trade on differences in expectations. Therefore, relaxing margin requirements means an increased speculative activity and hence volatility increases.

In our models, beliefs are an important factor. Under the rational beliefs principle agent 1 is sometimes optimistic and sometimes pessimistic. In optimistic states he tries to gain from investing into the risky asset while in pessimistic states he is interested in wealth preservation. With a looser margin requirement the agents can better preserve their capital in pessimistic states and thus consumption volatility declines. However, they are still trading on differences in beliefs. Therefore, if margin requirements are strict the speculative effect on volatility dominates and therefore volatility increases if margin requirements are relaxed. While if margin requirements are already relaxed, the capital preservation becomes more important and thus volatility declines.

3.3.2 Collateral Premium

[10]One reason for these varying result may be that there is not enough variation in margin requirements. Margin requirements in the US are set by the Federal Reserve and has not been changed since 1974.

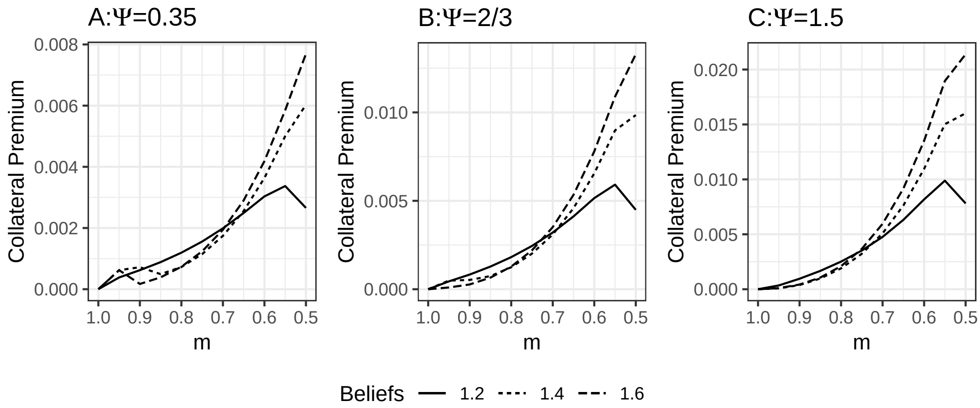

Figure 7: The Collateral Premium as a function of the margin requirement m. The initial wealthshare is $\omega^1 = 0.5$. The coefficient of relative risk-aversion is $\gamma = 1.5$ and the Elasticity of Intertemporal substitution is $\psi = 1.5$.

Following Fostel and Geanakoplos (2008), we define the collateral premium as follows:

$$CV^h = E_\Gamma \left[\frac{\mu^{hc}(1-m)\theta^h \min(q^+ + d^+)}{q\lambda^{hb}} \right], \tag{39}$$

and λ^{hb} is the lagrange-multiplier associated with the budget constraint of agent h. The interpretation of the collateral value is straight forward and is the fraction of the assets' value that is due to it's use as collateral. If agents cannot borrow, i.e. $m = 1$, then there is no value added through its use as collateral and the collateral premium is zero.

Figure 7 shows the collateral premium of the risky asset for the three cases of beliefs as well as the three cases of EIS. We can see that, in general, more disagreement leads to a higher collateral premium, regardless of the EIS, as the margin constraint is now binding more often. Similarly, with a larger elasticity of intertemporal substitution agents have a more risky portfolio and therefore the margin constraint is also more often binding. Thus, a larger EIS implies a larger collateral premium

However, as it can also be seen the Collateral Premium is rather small, i.e. even at the largest it is only about 2% of the assets value.

4 Conclusion

In this paper we studied the impact of market completeness and elasticity of intertemporal substitution on survival and the impact on asset prices in an economy with heterogeneous beliefs. Simulation results have shown that agents who do not believe that the empirical distribution is the true underlying distribution survive on the long-run. Additionally the evolution of financial wealth in the economy depends on the ability to borrow as well as the elasticity of intertemporal substitution.

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A Definitions

For the definition of stability and ergodicity use the definitions from Kurz (1994).

Let Ω denote a sample space, $\mathcal{F}$ a σ-field of subsets of Ω, T the shift transformation such that $T(x_t, x_{t+1}, x_{t+2}, ...)$ and Π a probability measure. Define now

$$1_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases} . \tag{40}$$

The relative frequency of the set S visited by the dynamical system given that it start at x as follows

$$m^n(S)(x) = \frac{1}{n}\sum_{k=0}^{n-1} 1_S\left(T^k x\right). \tag{41}$$

Then we define stability and ergodicity as follows

Definition 4 (Stability)**.** *A dynamical system $(\Omega, \mathcal{F}, T\Pi)$ is said to be stochastically stable if for all cylinders $Z \in \mathcal{F}$ the limit of m^n exist Π a.e., and the limit is denoted by*

$$\tilde{m}(S)(x) = \lim_{n\to\infty} m^n(S)(x). \tag{42}$$

Definition 5 (Invariance)**.** *$S \in \mathcal{F}$ is said to be invariant with respect to T if $T^{-1}S = S$. A measurable function is said to be invariant with respect to T if for any $x \in \Omega$, $f(T(x)) = f(x)$.*

Definition 6 (Ergodicity)**.** *A dynamical system is said to be ergodic if $\Pi(S) = 0$ or $\Pi(S) = 1$ for all invariant sets S.*

Definition 7 (Compatibility with the Data)**.** *We say that a probability $Q \in \mathcal{P}(\Omega)$ is compatible with the data if*

(a) $(\Omega, \mathcal{F}, Q, T)$ is stable with a stationary measure m. That is, for all cylinders

$S \in \mathcal{F}$

$$m_Q(S) \overset{d}{=} \lim_{n\infty} \frac{1}{n} \sum_{k=0}^{n-1} Q(T^{-k}S) = m(S)$$

(b) Q satisfies the tightness condition Π.

B Derivation of the first order conditions

For ease of notation, we drop the reference to a household h. The maximization problem of the agent can be written as the following Lagrangian:

$$\begin{aligned}
\mathcal{L} &= \left((1-\beta)\,(c_t)^{\frac{1-\gamma}{\rho}} + \beta[E_{Q_t}\left[(U_{t+1}g_{t+1})^{1-\gamma}\,|\mathcal{F}_t\right]^{\frac{1}{\rho}}\right)^{\frac{\rho}{1-\gamma}} && (43)\\
&\quad -\lambda_t^b\left(c_t + \theta_t q_t + b_t p_t - \theta_{t-1}(q_t + d_t) - \frac{b_{t-1}}{g_t} - e_t\right) - \lambda_t^s \theta_t^h \\
&\quad -\lambda_t^c\left(b_t + (1-m)\theta_t \min_{s^{t+1}|s^t}(q_{t+1} + d_{t+1})\right).
\end{aligned}$$

The lagrange multiplier with respect to the budget constraint is denoted by μ_t^b, for the short-sale constraint μ_t^s and for the collateral constraint μ_t^c. Taking now the derivative with respect to consumption and rearranging yields

$$\frac{\partial \mathcal{L}}{\partial c_t} = (U_t)^{\psi^{-1}}\, c_t^{-\psi^{-1}} = \lambda_t^b. \tag{44}$$

The derivative with respect to asset purchases is

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \theta_t} &= (U_t)^{\psi^{-1}}\, \beta E_{Q_t}\left[(U_{t+1}g_{t+1})^{1-\gamma}\right]^{\frac{1-\rho}{\rho}} E_{Q_t^h}\left[(U_{t+1}g_{t+1})^{-\gamma}\, g_{t+1}\frac{\partial U_{t+1}}{\partial \theta_t}\right] && (45)\\
&\quad -\lambda_t^b q_t - \lambda_t^s - \lambda_t^c(1-m)\min_{s^{t+1}|s^t}(q_{t+1} + d_{t+1}),
\end{aligned}$$

and because of the envelope theorem the derivative of U_{t+1} with respect to

θ_t is given by

$$\frac{\partial U_{t+1}}{\partial \theta_t} = \frac{\partial U_{t+1}}{\partial c_{t+1}} \frac{\partial c_{t+1}}{\partial \theta_t} = (U_{t+1})^{\psi^{-1}} (1-\beta)(c^h_{t+1})^{-\psi^{-1}} (q_{t+1} + d_{t+1}). \tag{46}$$

Combining the last two equations we get

$$q_t \lambda^b_t = (U_t)^{\psi^{-1}} \beta E_{Q_t} \left[(U_{t+1} g_{t+1})^{1-\gamma} \right]^{\frac{1-\rho}{\rho}} \tag{47}$$

$$E_{Q_t} \left[(U_{t+1})^{\psi^{-1}-\gamma} g^{1-\gamma}_{t+1} (1-\beta) \left(c^h_{t+1} \right)^{-\psi^{-1}} (q_{t+1} + d_{t+1}) \right] \tag{48}$$

$$+\lambda^s + \lambda^c (1-m) \min_{s^{t+1}|s^t} (q_{t+1} + d_{t+1}).$$

The first order conditions for bond holdings can be derived similarly, i.e.

$$p_t \lambda^b_t = (U_t)^{\psi^{-1}} \beta E_{Q_t} \left[(U_{t+1} g_{t+1})^{1-\gamma} \right]^{\frac{1-\rho}{\rho}} \tag{49}$$

$$E_{Q_t} \left[(U_{t+1})^{\psi^{-1}-\gamma} g^{-\gamma}_{t+1} (1-\beta) \left(c^h_{t+1} \right)^{-\psi^{-1}} \right] \tag{50}$$

$$+\lambda^c.$$

C Numerical Algorithm

To solve for the stationary equilibrium we use a time-iteration algorithm. The algorithm proceeds as follows:

Step 0: Set an error-tolerance ϵ and form a grid M over $[0,1]$, Set an initial guess f^0 for policy and price functions.

Step 1: Given a set of policy and price functions f^{n-1}, we obtain a new set of policies and prices f^n by solving the system of equilibrium conditions, the law of motion for the wealth share and optimality conditions (51)-(59) for each gridpoint $(\omega, y) \in M \times \mathcal{Y}$. As the short-sale constraint as well as the margin requirement are not always binding, the Lagrange-Multipliers λ^{hc} and λ^{hs} are not differentiable at edge-cases. Hence, the system of equation is not differentiable. To circumvent the problem we use the Garcia-Zangwill trick (Zangwill and Garcia (1981)) and replace the lagrange multiplier λ^{hs} and

λ^{hc} with $\lambda^{hs+} = \max\{0, \lambda^{hs}\}^2$, $\lambda^{hs-} = \max\{0, -\lambda^{hs}\}^2$, $\lambda^{hc+} = \max\{0, \lambda^{hc}\}^2$, $\lambda^{hc-} = \max\{0, -\mu^{hc}\}^2$. Thus, the system of equations is now as follows:

$$\begin{aligned}
q_n (U^h)^{\psi^{-1}} (c_n^h)^{-\psi^{-1}} &= (U_n^h)^{\psi^{-1}} \beta E_Q^h \left[\left(U_{n-1}^h(\omega^+, y^+) g(y^+) \right)^{1-\gamma^h} \right]^{\frac{1-\rho^h}{\rho^h}} \\
&\quad E_Q^h \Bigg[\left(U_{n-1}^h(\omega_{n-1}^+, y^+) \right)^{(\psi^h)^{-1} - \gamma^h} c_{n-1}^h(\omega^+, y^+)^{-\psi^{-1}} \\
&\quad (1-\beta) g(y^+)^{1-\gamma^h} (q_{n-1}(\omega^+, y^+) + d(y^+)) \Bigg] \\
&\quad + \lambda_n^{hs+} + \lambda_n^{hc+} (1-m) \min_{y^+} (q_{n-1}(\omega^+, y^+) + d(y^+)) \qquad (51) \\
p_n (U_n^h)^{\psi^{-1}} (c_n^h)^{-\psi^{-1}} &= (U_n^h)^{\psi^{-1}} \beta E_Q^h \left[\left(U_{n-1}^h(\omega^+, y^+) g(y^+) \right)^{1-\gamma^h} \right]^{\frac{1-\rho^h}{\rho^h}} \\
&\quad E_Q^h \Bigg[\left(U_{n-1}^h(\omega^+, y^+) \right)^{(\psi^h)^{-1} - \gamma^h} c_{n-1}^h(\omega^+, y^+)^{-\psi^{-1}} \\
&\quad (1-\beta) g(y^+)^{-\gamma^h} \Bigg] + \lambda^{hc+} \qquad (52) \\
c_n^h &= e^h + \omega^h (q_n + d) - \theta^h q_n - b^h p_n \qquad (53) \\
b_n^1 + b_n^2 &= 0 \qquad (54) \\
\theta^1 + \theta^2 &= 1 \qquad (55) \\
c_n^1 + c_n^2 &= 1 \qquad (56) \\
\omega_n^{h+} &= \frac{\theta^h (q_{n-1}(\omega_n^+, y^+) + d(y^+)) + \frac{b_n^h}{g(y^+)}}{q_{n-1}(\omega^+, y^+) + d(y^+)} \qquad (57) \\
\lambda_n^{hs-} &= \theta_n^h \qquad (58) \\
\lambda_n^{hc-} &= \left(b_n^h + \theta_n^h (1-m) \min_{y^+} (q_{n-1}(\omega^+, y^+) + d(y^+)) \right) \qquad (59)
\end{aligned}$$

Here, equations (51) and (52) are the first order conditions for asset and bond holdings respectively. Equation (53) is the budget constraint while equations (54)-(56) are the equilibrium conditions, equation (57) is the dynamics for wealthshare and equations (58) and (59) are the modified complementary slackness conditions.

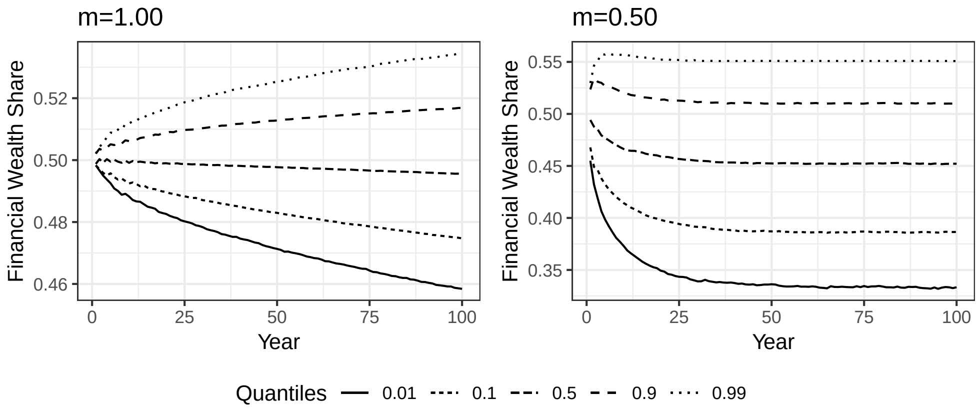

Figure 8: This graph shows the dynamics of the wealth distribution of Agent 1 over 100 years for model 2. The quantiles of the wealth distribution are $0.01, 0.1, 0.5, 0.9, 0.99$. The initial wealthshare is $\omega^1 = 0.5$. The coefficient of relative risk-aversion is $\gamma = 1.5$ and the Elasticity of Intertemporal substitution is $\psi = 0.35$. The margin requirements are $m = 1.00$ and $m = 0.50$.

To solve for equilibrium prices, in addition to next periods prices, only next periods consumption and Value-function are needed and not portfolio choices. Thus, we do not need to interpolate next periods portfolio choices.

Step 2: Prices and policy functions are updated until $||f^n - f^{n-1}|| < \epsilon$.

In our application, the grid M has 101 equidistant points and ϵ is set to 10^{-4} and the algorithm is implemented in Matlab. To solve the system of nonlinear equations (51)-(59) one can use a nonlinear-equation solver such as fsolve. For our application, we set ϵ to $1e - 4$ and the number of gridpoints is 101.

D Additional Results

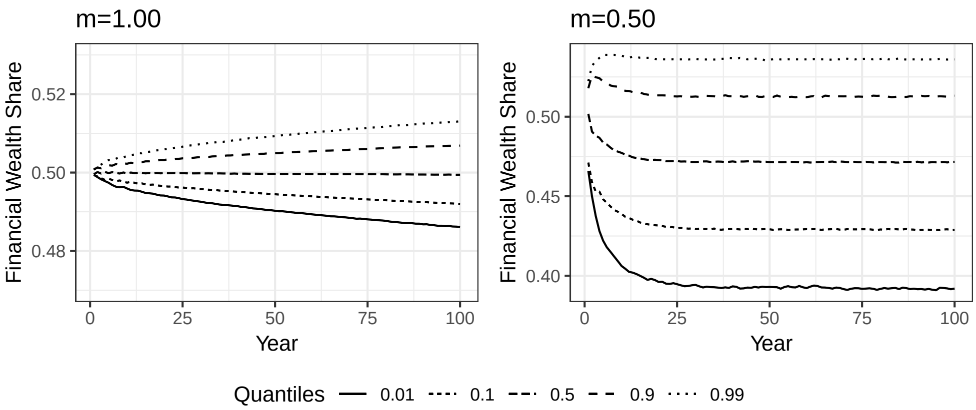

Figure 9: This graph shows the dynamics of the wealth distribution of Agent 1 over 100 years for model 2. The quantiles of the wealth distribution are $0.01, 0.1, 0.5, 0.9, 0.99$. The initial wealthshare is $\omega^1 = 0.5$. The coefficient of relative risk-aversion is $\gamma = 1.5$ and the Elasticity of Intertemporal substitution is $\psi = 0.66$. The margin requirements are $m = 1.00$ and $m = 0.50$.

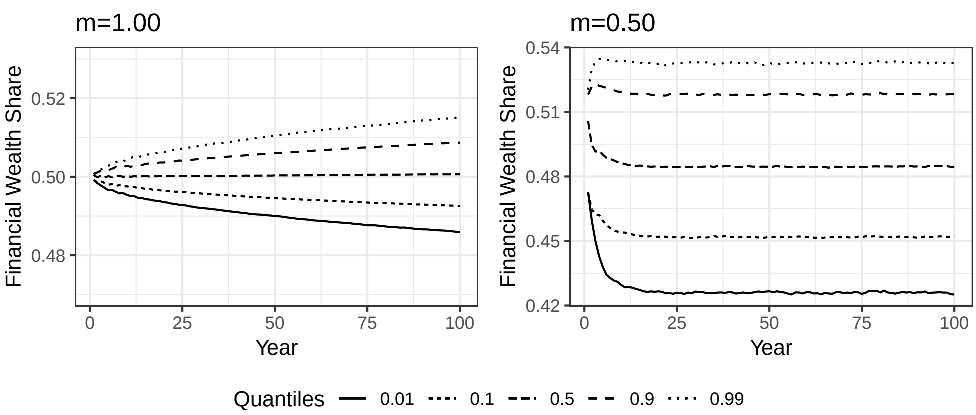

Figure 10: This graph shows the dynamics of the wealth distribution of Agent 1 over 100 years for model 2. The quantiles of the wealth distribution are $0.01, 0.1, 0.5, 0.9, 0.99$. The initial wealthshare is $\omega^1 = 0.5$. The coefficient of relative risk-aversion is $\gamma = 1.5$ and the Elasticity of Intertemporal substitution is $\psi = 1.50$. The margin requirements are $m = 1.00$ and $m = 0.50$.

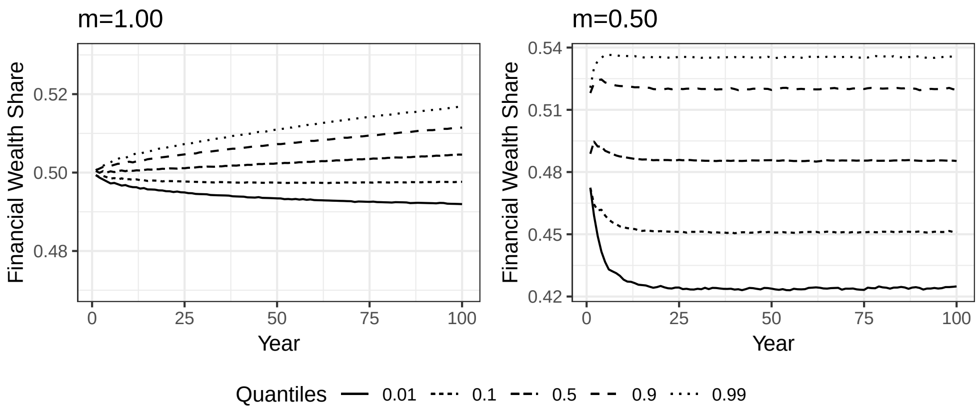

Figure 11: This graph shows the dynamics of the wealth distribution of Agent 1 over 100 years for model 3. The quantiles of the wealth distribution are $0.01, 0.1, 0.5, 0.9, 0.99$. The initial wealthshare is $\omega^1 = 0.5$. The coefficient of relative risk-aversion is $\gamma = 1.5$ and the Elasticity of Intertemporal substitution is $\psi = 1.50$. The margin requirements are $m = 1.00$ and $m = 0.50$.

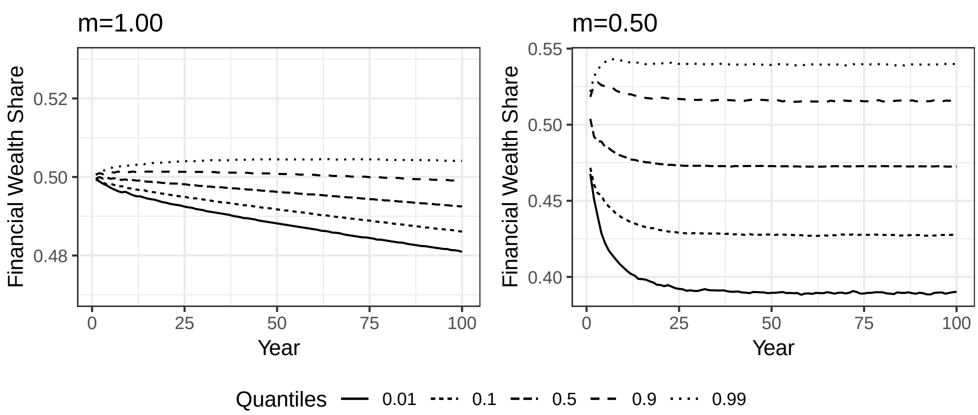

Figure 12: This graph shows the dynamics of the wealth distribution of Agent 1 over 100 years for model 3. The quantiles of the wealth distribution are $0.01, 0.1, 0.5, 0.9, 0.99$. The initial wealthshare is $\omega^1 = 0.5$. The coefficient of relative risk-aversion is $\gamma = 1.5$ and the Elasticity of Intertemporal substitution is $\psi = 2/3$. The margin requirements are $m = 1.00$ and $m = 0.50$.

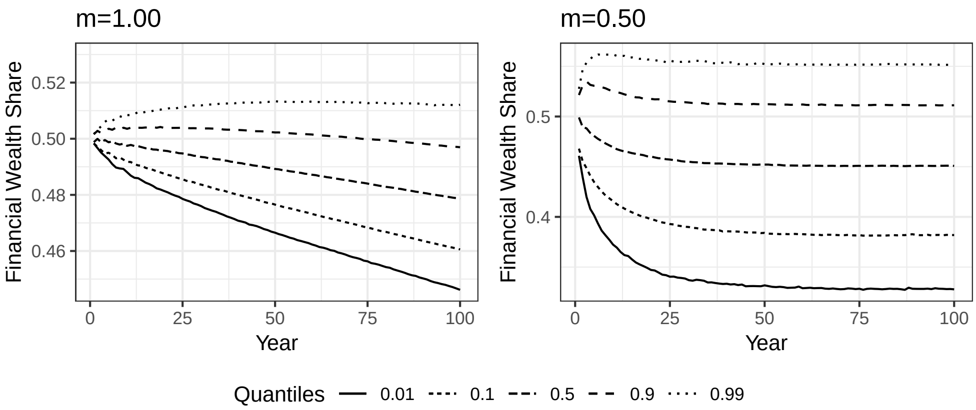

Figure 13: This graph shows the dynamics of the wealth distribution of Agent 1 over 100 years for model 3. The quantiles of the wealth distribution are $0.01, 0.1, 0.5, 0.9, 0.99$. The initial wealthshare is $\omega^1 = 0.5$. The coefficient of relative risk-aversion is $\gamma = 1.5$ and the Elasticity of Intertemporal substitution is $\psi = 0.35$. The margin requirements are $m = 1.00$ and $m = 0.50$.