Financial Innovation and Asset Prices with Heterogeneous Beliefs

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Abstract

This paper studies the impact of financial innovation on an asset price dynamics in an exchange economy witht recursive preferences and heterogeneous beliefs. To model heterogeneous beliefs we restrict the set of possible beliefs to the subset of rational beliefs in the sense of Kurz (1994). In our model financial innovations are interpreted as additional financial securities whose payoff depends on the aggregate state of the economy. Simulation results indicate that additional financial securities increase the riskyness of agents' portfolios and in turn increase the volatility of the Price/Dividend Ratio.

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1 Introduction

In a modern economy financial innovations[1] are inevitable.[2] For instance, since the seminal contribution of Black and Scholes (1973) and Merton (1973) the market for derivatives, in particular options, has increased exponentially. Figure 1 shows the development of exchange traded futures and options between 1993 and 2015 and we see that the notional value of outstanding options and futures has more than quadrupled from the early nineties to its peak prior to the global financial crisis. Another important financial innovation has been securitization, in particular collateralized debt obligations (CDO) and mortgage backed securities (MBS) backed by non-prime loans. The market for these products has grown exponentially in the early 2000s until it reached its peak in 2007.

The classical view emphasizes the role of financial markets as a tool for sharing risks. According to this view financial innovations improve consumption smoothing and thus lower volatility in the economy (Allen and Gale (1994) or Shiller (1994)). The rapid development of financial innovations has not been uncontroversial and critics highlight the downsides of financial innovations and argue that innovations in financial markets do not necessarily provide benefits to the economy. For instance, financial institutions may use innovative financial products to exploit investors' misunderstandings of financial markets (Henderson and Pearson (2011)). Another point critics of financial innovations make is that they destabilize financial markets and the economy (Rajan (2006))

[1] We emphasize that we are talking about creation of new financial assets and not about changes in the delivery of financial services using modern information technology which is also known as 'FinTech'.

[2] An overview of financial innovations up to the mid 1980s is given by Matthews (1994)

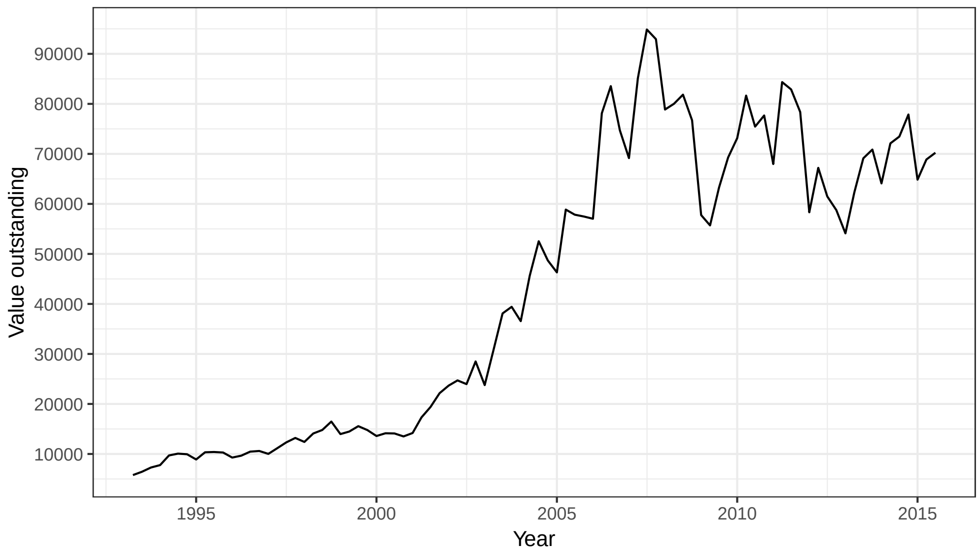

Figure 1: Figure showing the notional value outstanding of exchange traded options and futures (source: Bank for International Settlements). Amounts are in billions of dollars.

and that financial innovations were at the heart of the global financial crisis in 2007-2009.

This paper studies the relationship between financial innovation and volatility of financial markets. In order to understand this relationship we use a dynamic general equilibrium model with heterogeneous beliefs and recursive preferences in which agents can trade in up to three assets: One real asset which represents a claim to aggregate output and two financial assets. While the net-supply of the real asset is normalized to one thee net-supply of the financial assets is zero. One of these financial assets is a risk-free bond while the payoff of the second financial asset depends on the state and is therefore not risk-free. This additional financial asset represents financial innovation in our model. Furthermore, we do not allow for arbitrary beliefs of agents and beliefs of agents are restricted to the set of rational beliefs in the sense of Kurz

(1994). Under the rational beliefs principle, agents have time-varying heterogeneous beliefs but are said to be rational because the unconditional distribution of beliefs has to be the same as the unconditional distribution of observable variables in the economy.

The core mechanism of our model is the endogeneous distribution of financial wealth across agents which is determined by the speculative motive and the precautionary savings motive. The speculative motive implies that agents who bet in the wrong direction are punished in the sense that their financial wealth depletes. Adding an additional financial asset expands the possibilities to speculate and thus volatility of financial wealth increases and in turn volatility of asset prices. However, the literature on survival in financial markets emphasizes the role of precautionary savings as a mechanism that affects the distribution of financial wealth (see f.i. Cogley, Sargent, and Tsyrennikov (2013) or Borovicka (2016)), i.e. agents with a stronger motive for precautionary savings are less likely to be driven out of the market. As financial innovations affect the equilibrium interest rate they also affect the distribution of financial wealth via precautionary savings. However, under our parameterization of the model the speculative motive always dominates and volatility of financial wealth increases and therefore also the volatility of asset prices.

Literature

Our paper is related to several strands of literature. First, it is related to the vast literature on financial innovation. An early contribution was made by Hart (1975) who showed by example that introducting additional assets to complete the market may be decrease welfare in the economy. The decrease in welfare stems from price effects of financial innovations, i.e. the introduction of a new asset may change relative prices of existing assets and therefore agents may choose a different portfolio which subsequently affects consumption. A general result in this strand of literature is that financial innovations may arbitrary effects on welfare (Cass and Citanna (1998)). The effects on portfolio risk of financial innovations have been studied by Simsek (2013a) and Simsek (2013b) who argues that financial innovations have two opposing effects. While financial innovations may improve risk-sharing, they may also enable speculation and thus portfolio risk may decrease or increase.

The effects of introducing derivatives on underlying assets has been studied by Detemple and Selden (1991), who show in a mean-variance model that the introduction of options increases the price of an underlying asset. Zapatero (1998) shows in an economy with heterogeneous beliefs that financial innovation increases volatility. Brock, Hommes, and Wagener (2009) come to a similar conclusion. Bhamra and Uppal (2007) show that financial innovation may increase volatility if the motive for precautionary savings is not too high. Iachan, Nenov, and Simsek (2015) argue that financial innovation contributed to declining asset returns. Fostel and Geanakoplos (2012) study the effects of financial innovation in a leverage economy and argue that financial innovation may contribute to bubbles. A closely related paper is Buss, Uppal, and Vilkov (2017) who study an economy with two Lucas trees and heterogeneous beliefs. In their model financial innovation is interpreted as removing the participation constraint on the second tree. Our model differs in one important dimension. They interpret financial innovation as relaxing the participation constraint on the second tree while we interpret financial innovation as creating a new financial asset.

This paper is connected to the literature on asset prices with heterogeneous beliefs and portfolio constraints.Harrison and Kreps (1978) initiated the literature on asset prices with heterogeneous beliefs. He shows that in the presence of short-sale constraints the price of an asset can rise above its fundamental value due to a speculative premium. Gallmeyer and Hollifield (2008) study the impact of short-sale constraints on asset prices and argues that the effect on asset prices and volatility depends on the elasticity of intertemporal substitution. Other studies include Detemple and Murthy (1994) or Chabakauri (2015) . These models usually have fixed set of financial and real assets while in our model the set of financial assets changes.

2 The Model

In this section we describe the model that is used to study the effects of financial innovation. Section 2.1 and 2.2 describes the general setup and the equilibrium conditions of the economy with heterogeneous beliefs. In Section 2.3 we put more structure on set of possible beliefs and restrict the set of possible beliefs to the subset of Rational Beliefs in the sense of Kurz (1994).

2.1 Setup

We consider an endowment economy with a single consumption good in infinite horizon. Time runs from $t = 0$ to ∞. There are H types of consumers:

$$h \in \mathcal{H} = \{1, 2, ..., H\}$$

in the economy. These consumers might differ in many dimensions including their preferences and their endowment of final good e_t^h. Consumers might also differ in their initial endowment of a real asset that pays off real dividends. However, in this paper we focus on the heterogeneity of beliefs over the evolution of the exogeneous states of the economy. There are S possible exogeneous states:

$$s \in \mathcal{S} = \{1, 2, ..., S\}.$$

The state captures both aggregate uncertainty (e.g. dividends) and idiosyncratic shocks. The evolution of the economy is captured by the realizations of the shocks over time:$s^t = (s_0, s_1, ..., s_t)$. We assume that the shocks follow a markov-process with the transition probabilities $\pi(s, s')$.

Real asset: There is one real asset in the economy that pays off state-dependent real dividends $d(s_t)$. Agents can purchase $\theta_t^h = \theta^h(s^t)$ units of the assets, which can also be used as collateral for borrowing. The ex-dividend price of the asset in history s^t is denoted by $q_t = q(s^t)$. Consumers are also not allowed to short sell the asset.[3] Furthermore, the total supply of the real asset

[3]Giménez (2003) shows that introducing short-sale constraints may lead to a multiplicity of equilibria. However, for the simulation we proceed as if there is a unique equilibrium.

is 1.

Financial assets: In addition to the real asset, we assume that households in the economy can trade in several financial assets. These financial assets are in zero net-supply. We denote the set of tradable financial assets as $\mathcal{J}$. The payoff of a financial asset $j \in \mathcal{J}$ is denoted by $f_{j,t}$, the price of it is denoted by $p_{j,t}$. Agent $h \in \mathcal{H}$ can buy or sell $\theta^h_{j,t}$ units of the financial asset. The seller of a financial asset has to put up enough collateral to cover the payoffs of all financial asset. Thus, we have the following collateral constraint for the financial assets:

$$\sum_{j \in \mathcal{J}} [\theta^h_{j,t}]^- f_{j,t+1} \geq -m \min \theta^h_t (q_{t+1} + d_{t+1}). \tag{1}$$

With $[x]^- = min(0, x)$ and $[x]^+ = max(0, x)$. The collateral constraint implies that if agents issue financial assets they can only use the real asset as collateral. We also treat sales and purchases of financial assets separately, i.e.

$$\theta^h_{j,t} = [\theta^h_{j,t}]^+ + [\theta^h_{j,t}]^-. \tag{2}$$

Consumers: Before we state the consumers' problem we make the following assumptions about the consumers:

Assumption 1. 1. *Each agent believes the economy is Markovian.*

2. *Each agent believes that no single agent can affect the equilibrium.*

As most stochastic processes can be restated as a markov-process the first assumption is not restrictive. The second assumption simply says that households are pricetakers.

In each state s^t, each consumer is endowed with some endowment $e_t^h = e^h(s_t)$ units of the consumption good. The aggregate endowment of the economy with the consumption is given by $\bar{e}_t = \sum_{h \in \mathcal{H}} e_t^h + d_t$ and the growth rate is denoted by $g_t = \frac{\bar{e}_t}{\bar{e}_{t-1}}$. Furthermore, weassume that they have recursive preferences as in Epstein and Zin (1989) and Weil (1990) Consumers take the sequence of prices $\{p_t, q_t\}$ as given and maximize the following recursive utility functoion:

$$U_t^h = \left((1-\beta) \left(c_t^h \right)^{\frac{1-\gamma^h}{\rho^h}} + \beta \left[E_{Q_t^h} \left[\left(U_{t+1}^h \right)^{1-\gamma^h} | \mathcal{F}_t \right]^{\frac{1}{\rho^h}} \right) \right)^{\frac{\rho^h}{1-\gamma^h}}, \tag{3}$$

with β as the subjective discount factor, γ^h asthe coefficient of relative risk aversion, and the Elasticity if Intertemporal Substitution $\psi^h \geq 0$. The parameter ρ^h is defined as $\rho^h := (1-\gamma^h)/(1-\frac{1}{\psi^h})$. And Q_t^h represents the subjective beliefs of agent h subject to the information set $\mathcal{F}_t$. The maximization problem is subject to the normalized intertemporal budget constraint:

$$c_t^h + q_t\theta_t^h + \sum_{j \in \mathcal{J}} \left([\theta_{j,t}^h]^- p_{j,t} + [\theta_{j,t}^h]^+ p_{j,t} \right) \leq e_t^h + \sum_{j \in \mathcal{J}} \left([\theta_{j,t-1}^h]^+ + [\theta_{j,t-1}^h]^{-1} \right) f_j^h{}_{t-1} + (q_t + d_t)\theta_{t-1}^h, \tag{4}$$

the collateral constraint

$$\sum_{j \in \mathcal{J}} [\theta_{j,t}^h]^- f_{j,t+1} \geq -m \min \theta_t^h (q_{t+1} + d_{t+1}) \tag{5}$$

and the short-sale constraint on the real asset

$$\theta_t^h \geq 0 \tag{6}$$

The descriprion of the optimization problem of the consumers implies that for consumer h in period t the demand correspondence depends on the portfolio choices made in previous, the current state s_t and her beliefs $\mathcal{Q}_t^h$:

$$\theta_t^h = \theta^h(\theta_{t-1}^h, \sum_{j\in\mathcal{J}} \theta_{j,t-1}^h, s_t, \mathcal{Q}_t^h) \tag{7}$$

$$\theta_{j,t}^h = \theta_j^h(\theta_{t-1}^h, \sum_{j\in\mathcal{J}} \theta_{j,t-1}^h, s_t, \mathcal{Q}_t^h) \quad \forall j \in \mathcal{J} \tag{8}$$

2.2 Equilibrium

For all periods $t = 1, ...$ the markets clear, i.e.

- The market for the real asset clears

$$\sum_{h\in\mathcal{H}} \theta_t^h = 1 \quad \forall t = 1, 2, 3, ... \tag{9}$$

- The market for all financial assets clear

$$\sum_{h\in\mathcal{H}} \left([\theta_{j,t}^h]^- + [\theta_{j,t}^h]^+ \right) = 0 \quad \forall j \in \mathcal{J} \text{ and } \forall t = 1, 2, 3, ... \tag{10}$$

- The market for the consumption good clears

$$\sum_{h\in\mathcal{H}} c_t^h = \sum_{h\in\mathcal{H}} e_t^h + d_t \quad \forall t = 1, 2, 3, ... \tag{11}$$

Together with the demand correspondences the market clearing conditions imply that equilibrium prices in period t are a function of endogeneous state variables, the exogeneous state, and the distribution of beliefs. Thus, we have the following equilibrium mapping:

$$\begin{bmatrix} q_t \\ f_{j,t} \end{bmatrix} = f\left(\left(\theta^h_{t-1}, \left(\theta^h_{j,t-1}\right)_{j\in\mathcal{J}}\right)_{h\in\mathcal{H}}, s_t, \left(Q^h_t\right)_{h\in\mathcal{H}}\right) \tag{12}$$

2.3 The Structure of Beliefs

So far, we have taken the beliefs $(Q^h_t)_{h\in\mathcal{H}}$ as given and we are now describing the construction of rational beliefs. The key tool in the construction of rational beliefs is the introduction of generating variables which fully describe the beliefs of the agents. Then the conditional stability theorem by Kurz and Schneider (1996) states that our construction constitutes a rational belief.

First, we describe the construction of the generating variables. Let X denote the state-space of data and observable $(s_t, p_t, q_t, (\theta^h_t, b^h_t, c^h_t)_{h\in\mathcal{H}})$ for all t and X^∞ the state space for the entire sequence. The Borel σ field generated by X^∞ will be denoted as $\mathcal{B}(X^\infty)$. The true stochastic process of the economy is described by a stochastic dynamic system $(X^\infty, \mathcal{B}(X^\infty), T, \Pi)$, where T denotes the shift-transformation[4] and Π the true probability measure.

We define now a rational belief:

Definition 1. *(Rational Beliefs) A sequence of effective beliefs* $\{Q^h_t\}^\infty_{t=0}$ *are a rational belief if the sequence is stable and ergodic, is compatible with the data and it induces a*

[4]The shift transformation T is defined as $x_{t+1} = Tx_t$. It is not assumed to be invertible, i.e. $T^{-1}x_{t+1} \neq x_t$, which implies that any future evolution is not associated with a unique past.

stationary measure that is equivalent to the one induced by the empirical measure Π.

The crucial part of this definition is that rational beliefs have to be 'compatible with the data'[5]. Intuitively the long-run distribution of the agents' beliefs has to be the same as the long-run distribution of prices and sates. Or, in other words it states that unconditional beliefs have to coincide with the empirical measure. However, rational beliefs still allows for mistakes as the definition does not require the belief to the true probability. It is important to note the rational beliefs principle rules out fixed (or dogmatic) beliefs, unless they believe that the empirical distribution is the true distribution. Do note that this definition of rational beliefs does not require agents to know the equilibrium map (12), instead agents deduce the relationships between variables on the observable data.

However, the definition of rational beliefs in this sense that does not tell us how we should construct rational beliefs or how agents should learn from the available data and it merely restrict the set of possible beliefs. Furthermore, a belief on X^∞ is a rather complicated object and it may prove impossible to check stability. Hence, instead modelling the learning process we pose the problem differently: Given the dynamic system $(X^\infty, \mathcal{B}(X^\infty), T, \Pi)$ we construct a sequence of effective beliefs that are rational beliefs.

To include the beliefs of the agents we follow Kurz and Schneider (1996) expand the probability space and include the sequence of generating variables $(n_t^h)_{t=1}^\infty$. Now, agent h forms a belief Q_t^h on $\left(\left(X \times \mathcal{N}^h\right)^\infty, \mathcal{B}\left(\left(X \times \mathcal{N}\right)^\infty\right)\right)$, where $\mathcal{N}^h := \{0, 1\}$ denotes the state space of n_t^h, and $\mathcal{B}\left(\left(X \times \mathcal{N}^h\right)^\infty\right)$ is the Borel σ-field generated by $\left(X \times \mathcal{N}^h\right)^\infty$. Now let $n^{ht} := (n_1^h, n_2^h, ..., n_t^h)$, i.e. the

[5]We provide a formal definition in Appendix A

history of generating variables n_t^h up to period t. Then, each finite history n^{ht} determines agent $h's$ effective belief in period t denoted by $Q_t^h(A) = Q^h(A|n^{ht})$ for $A \in \mathcal{B}(X^\infty)$, which is a probability measure on $(X^\infty, \mathcal{B}(\Sigma^\infty))$. The analysis is simplified by the following assumption:

Assumption 2. *The marginal distribution for n_t^h with respect to Q_t^h is i.i.d. with $Q^h(n_t^h = 1) = \mu^h$.*

Assumptions 1 and 2 imply that the effective belief Q_t^h is solely determined by the generating variable n_t^h, i.e. $Q_t^h(A) = Q^h(A|n_t^h)$ for $A \in \mathcal{B}(\Sigma^\infty)$. Hence, we interpret the variable n_t^h as describing the state of belief of agent h in period t.

For example, the belief Q^h supports a regime switching model[6], then n_t^h describe the regime which agent h believes the economy is in. For instance, if $n_t^h = 1$ agent h may be optimistic about the economy while $n_t^h =$ corresponds to a pessimistic state of belief.

Furthermore, even though households switches between beliefs are i.i.d. the description above allows us to model a wide range of joint dynamics between beliefs and states.

In the next step we need to make sure that the conditional probabilities Q^h given $n^h \in (\mathcal{N}^\infty)$ are a rational belief. This is achieved via the Conditional Stability Theorem. However, before we state the Conditional Stability Theorem, we introduce some important notation.

Let Π_k^h denote the conditional probability of $\hat{\Pi}^h$ given a particular sequence

[6]Regime switching processes are popular in econometrics to model non-linearities in macroeconomic time series (e.g. Hamilton (1989))

of effective beliefs $k \in (\mathcal{N}^h)^\infty$:

$$\hat{\Pi}^h_k(\cdot) : (\mathcal{N}^h)^\infty \times \mathcal{B}(X^\infty) \mapsto [0,1]. \tag{13}$$

For each $A \in \mathfrak{B}(X^\infty)$, $\hat{\Pi}^h_k$ is a measurable function of k and for each k, $\hat{\Pi}^h_k(\cdot)$ is a probability on $(N^\infty, \mathcal{B}(X^\infty))$. For $A \in \mathcal{B}(X^\infty)$ and $B \in \mathcal{B}((\mathcal{N}^h)^\infty)$, we have

$$\hat{\Pi}^h(A \times B) = \int_{k \in B} \hat{\Pi}^h_k(A)\bar{\mu}^h(dk). \tag{14}$$

Also, as we noted above,

$$\Pi(A) = \hat{\Pi}^h(A \times (\mathcal{N}^h)^\infty), \forall A \in \mathcal{B}(X^\infty), \tag{15}$$
$$\bar{\mu}^h(B) = \hat{\Pi}(X^\infty \times B), \forall B \in \mathcal{B}((\mathcal{N}^h)^\infty).$$

If $(\Omega^h, \mathfrak{B}^h, T, \hat{\Pi}^h)$ is a stable dynamical system with a stationary measure $m^{\hat{\Pi}^h}$, we define the two marginal measures of $m^{\hat{\Pi}^h}$ as follows:

$$m(A) \quad := \quad m^{\hat{\Pi}^h}(A \times (\mathcal{N}^h)^\infty), \forall A \in \mathcal{B}(X^\infty). \tag{17}$$
$$m_{Q^h}(B) \quad := \quad m^{\hat{\Pi}}(X^\infty \times B), \forall B \in \mathcal{B}((\mathcal{N}^h)^\infty). \tag{18}$$

Also, let $\hat{m}_k$ denote the stationary measure of $\hat{\Pi}^h_k$, which is a measure on $(X^\infty, \mathfrak{B}(X^\infty))$. Given the construction of the dynamical system, we have the following theorem:

Theorem 1. *(Conditional Stability Theorem, Kurz and Schneider (1996)).*
Let $(\Omega^h, \mathcal{B}^h, T, \hat{\Pi}^h)$ be a stable and ergodic dynamical system. Then,

1. *$(X^\infty, \mathcal{B}^\infty, T, \hat{\Pi}^h_k)$ is stable and ergodic for $\hat{\Pi}^h$ a.a. k.*

2. *$\hat{m}_k^h$ is independent of k, $m_k^h = m = \Pi$.*

3. *If $(X^\infty, \mathfrak{B}(X^\infty), T, \hat{\Pi}_k^h)$ is stationary, then the stationary measure of $\hat{\Pi}_k^h$ is Π. That is*

$$\hat{m}_k^h = m = \Pi.$$

So far, our discussion on constructing rational beliefs did assume that agents do not know the equilibrium map (12). However, to simplify the computational model we assume that agents do know the equilibrium map (12)[7]. This implies that once agents have chosen their portfolios next periods prices depend only on the state s_t and the distribution of belief $(Q_t^h)_{h\in\mathcal{H}}$. This simplifies the construction of a computational model as agents need to form beliefs only over the exogeneous variables, i.e. they form beliefs over $\left(s_t, (n_t^h)_{h\in\mathcal{H}}\right)$

To illustrate and clarify the construction of rational beliefs, we consider an example similar to our simulation model discussed in a later section.

Consider an economy with two exogeneous states (e.g. high dividends and low dividends) and two agents. Both agents can be either optimistic in the sense that she assigns a higher probability on higher dividends then empirically observed or pessimistic in the sense that she assigns a lower probability on high dividends than empirically observed. Now, our state-space consists of 8 states. In particular, we have the tuple $\{d_t, n_t^1, n_t^2\}$. Now, in period t agents form beliefs not only over dividends but also over the distribution of future generating variables $\{n_{t+1}^1, n_{t+1}^2\}$.

This implies that the sequence of effective beliefs of a household Q_t^h must

[7]This assumption is similar to assumptions made in the literature on bayesian learning.

have the same stationary distribution as the tuple $\{d_t, n_t^1, n_t^2\}_{t=0}^{\infty}$. If the tuple has a Markov transition matrix Γ and the beliefs are represented by two transition matrices F_H^h, F_L^h the rationality condition implies that

$$\mu^h F_H^h + (1-\mu^h)F_L^h = \Gamma. \tag{19}$$

□

Now, using the generating variable, we can rewrite the portfolio choice (7)-(8) in terms of generating variables rather than beliefs Q_t^h:

$$\theta_t^h = \theta^h\left(\theta_{t-1}^h, b_{t-1}^h, p_t, q_t, s_t, n_t^h\right), \tag{20}$$

$$b_t^h = \theta^h\left(\theta_{t-1}^h, b_{t-1}^h, p_t, q_t, s_t, n_t^h\right). \tag{21}$$

From this, it follows that prices in the economy are given by

$$\begin{bmatrix} q_t \\ q_t^b \end{bmatrix} = f\left(\left(\theta_{t-1}^h, b_{t-1}^h\right)_{h\in H}, s_t, \left(Q_t^h\right)_{h\in\mathcal{H}}\right). \tag{22}$$

We define the stochastic primitives y_t as follows:

$$y_t = \left(s_t, (n_t^h)_{h\in\mathcal{H}}\right) \quad \forall t. \tag{23}$$

The state space of the stochastic primitives is now $\mathcal{Y}$. We assume that $\{y_t\}_{t=0}^{\infty}$ is a stable Markov process with a time homogeneous transition probability $P : \mathcal{Y} \to \mathcal{P}(\mathcal{Y})$, where $\mathcal{P}(\mathcal{Y})$ denotes the space of probability measures on $\mathcal{Y}$.

3 Quantitative Results

3.1 Parameterization

We assume that there are only 2 agents in the economy, that is, $H = 2$. We also assume that there are two growth states, i.e. $g_t \in \{\bar{g}, \underline{g}\}$.

The empirical distribution $\{g_t\}$ follows a markov-process:

$$\Psi = \begin{bmatrix} \phi & 1-\phi \\ 1-\phi & \phi \end{bmatrix}. \tag{24}$$

The stationary transition probability matrix for the tupel (g_t, n_t^1, n_t^2) has to satisfy the following conditions:

- the empirical distribution for the process g_t is specified by transition probability matrix Ψ.
- the marginal distribution for n_t^h is i.i.d with frequency of $\{n_t^h = 1\} = \alpha^h$.

Here, we use a specification similar to Kurz and Motolese (2001) as we know that the beliefs are compatible with the stationary distribution and it can generate large fluctuations. Furthermore, this specification allows for correlation between the three variables (g_t, n_t^1, n_t^2). We assume that the 8×8 matrix Γ has the following structure:

$$\Gamma = \begin{bmatrix} \phi A & (1-\phi)A \\ (1-\phi)A & \phi A \end{bmatrix}. \tag{25}$$

A is a 4×4 matrice defined by 6 parameters (α^1, α^2, a) and $a = (a^1, a^2, a^3, a^4)$

as follows:

$$A = \begin{bmatrix} a^1 & \alpha^1 - a^1 & \alpha^2 - a^1 & 1 + a^1 - \alpha^1 - \alpha^2 \\ a^2 & \alpha^1 - a^2 & \alpha^2 - a^2 & 1 + a^2 - \alpha^1 - \alpha^2 \\ a^3 & \alpha^1 - a^3 & \alpha^2 - a^3 & 1 + a^3 - \alpha^1 - \alpha^2 \\ a^4 & \alpha^1 - a^4 & \alpha^2 - a^4 & 1 + a^4 - \alpha^1 - \alpha^2 \end{bmatrix}. \tag{26}$$

We also have to specify the transition probability matrices that represent the beliefs of the agents. As noted above, agent $h \in \{1, 2\}$ in period t uses F_1^h when his generating variable is $n_t^1 = 1$ and F_2^h when his generating variable is $n_t^1 = 0$. The rationality of belief condition implies that

$$\alpha^h F_1^h + (1 - \alpha^h) F_2^h = \Gamma. \tag{27}$$

To fully pin down a traders' belief we only have to specify F_1^h while F_2^h can be inferred from Γ and F_1^h. The matrix F_1^h is parametrized by η^h as follows:

$$F_1^h(\eta^h) = \begin{bmatrix} \phi \eta^h A & (1 - \eta^h \phi) A \\ (1 - \phi) \eta^h A & (1 - (1 - \phi) \eta^h) A \end{bmatrix}. \tag{28}$$

From the above equation one can see that if $\eta^h > 1$ a trader places more weight on the growth states, i.e. he is overly optimistic that the economy grows when his beliefs are given by F_1^h. Furthermore, the larger the η^h implies a more optimistic trader. Furthermore, parameter α^h determines the frequency of optimistic beliefs, when $\alpha^h = 0.5$ then optimistic and pessimistic have the same frequency while $\alpha^h > 0.5$ implies that a trader is more often optimistic then pessimistic. This has also implications for pessimistic beliefs. In particular if $\eta^h > 1$ and $\alpha^h > 0.5$ then beliefs are more asymmetrically distributed to satisfy the rationality condition. In particular, a larger α^h means that agent h has more often optimistic beliefs about the state of the economy and to satisfy the rationality condition household h has to become even more pessimistic.

Following Mehra and Prescott (1985) we consider the following transition probability matrix for Ψ:

$$\Psi = \begin{bmatrix} 0.43 & 0.57 \\ 0.57 & 0.43 \end{bmatrix}, \tag{29}$$

and set $\overline{g} = 1.054$ and $\underline{g} = 0.982$. And in line with the literature we set the dividends d_t to $d_t = 0.15$.

For the beliefs of the agents we follow Kurz and Motolese (2001) and set $(a_1, a_2, a_3, a_4) = (0.5, 0.14, 0.14, 0.14)$. Furthermore, we assume that $\alpha^1 = \alpha^2 = \alpha = 0.57$. The maximum value for η is $1/0.57 \approx 1.7$ and we will examine several different cases of η. To focus on the survival aspect, we consider the case that agent 2 beliefs that the empirical distribution is the true distribution, i.e. $\eta^2 = 1$, while agent 1 does not believe that the empirical distribution is the true distribution. In particular we consider $\eta^1 \in \{1.2, 1.4, 1.6\}$.

Our choices for preferences follow the literature. We set the time-preference parameter to $\beta = 0.96$, the coefficient of relative risk-aversion is set to $\gamma = 1.5$ which is standard in the literature. On the other hand, for the value of the EIS there is a bigger range of estimates. Some authors estimate a rather low value for the EIS, for example Hall (1988) estimates a value much smaller than 1, while several asset pricing models (e.g., Collin-Dufresne, Johannes, and

Lochstoer (2014) or Bansal and Yaron (2004)) have used a EIS greater than 1. An EIS greater than 1 is needed to capture the negative correlation between consumption volatility and the price/dividend-ratio. For the baseline model we set the elasticity of intertemporal substitution for both agents to $\psi = 1.5$, a value which is in line with the asset pricing literature.

The model is solved using a policy function iteration and the details of the solution algorithm are outlined in appendix C. For the numerical algorithm, we collapse the three endogeneous state variables (holdings of real asset, the two financial assets) into one state variable, namely financial wealth share. The financial wealth W_t^h of a consumer is given by

$$W_t^h = \theta_{t-1}^h (q_t + d_t) + \sum_{j \in \mathcal{J}} \theta_{j,t-1}^h f_{j,t}. \tag{30}$$

Summing the financial wealth of all consumers and using equilibrium conditions the total financial wealth in the economy becomes $q_t + d_t$, i.e. the price and dividend of the real asset. Therefore, the financial wealth share w_t^h is

$$w_t^h = \frac{W_t^h}{\sum_{h \in \mathcal{H}} W_t^h} = \frac{\theta_{t-1}^h (q_t + d_t) + \sum_{j \in \mathcal{J}} \theta_{j,t-1}^h f_{j,t}}{q_t + d_t}. \tag{31}$$

Because of the collateral constraint the financial wealth share is bounded between 0 and 1.

The payoff of the second financial asset depends on the aggregate state. In particular we study two different types of financial assets, one that pays off if the economy expands (Model 1) and one that pays off when the economy is in recession (Model 2). These financial innovations are in addition to a bond, hence agents can trade in three assets - one real asset and two financial assets of which one is a bond and the payoff of the other financial asset is state-dependent. We compare these two types of financial innovations to a baseline economy with a bond only.

We do not study financial assets whose payoff depends on the individual state of belief for one reason. If payoffs depend on the individual state of beliefs it would require agents to reveal their true state of belief. However, agents do not have any incentives doing so and always pretend that their beliefs are in state where they have to pay the least. By conditioning the payoff on the observabele aggregate state we avoid this incentive-compatibility problem.

One may think that Model 1 and Model 2 are equivalent as pay offs of financial assets could potentially be replicated. For example, in Model 1 one could create a portfolio that has a pay off of 1 in the recessionary state by buying the bond and shorting the other financial asset. If the economy ends up in an expansionary state the consumer gets 1 consumption unit from the bond and has to pay 1 unit for selling the financial asset. Thus, the total pay off of the portfolio is 0 in expansionary states. Similarly, the total payoff of the portfolio is 1 in recessionary states. However, due to the collateral constraint the ability of consumers to create such replicating portfolios is limited.

3.2 Portfolio Choices

For the analysis of the portfolio choices we focus on the case of $\eta = 1.6$, i.e. there is a large disagreement between consumers about the growth of the economy. Figure 2 shows the distribution of bond and asset positions after 500

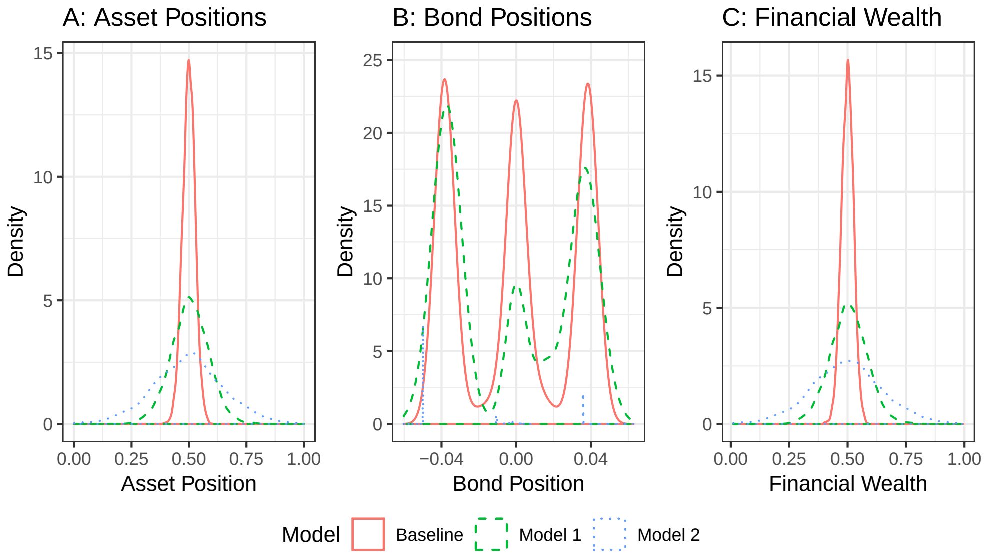

Figure 2: Portfolio Choices for Assets and Bonds and share of financial wealth.

years. Panel A shows the distribution of risky asset holdings and one can clearly see that the distribution of asset holdings clearly differs across the three economies. With the baseline economy having the narrowest distribution.

Panel B shows the bond positions and we see that for Model 2 there is no substantial trade in bonds implying that households prefer to trade in financial assets that pay off when the economy is in a recession. Furthermore, the distribution of asset holdings is the widest in Model 2. If consumers assign a high probability of being in a recession next period they are going to buy financial assets that pay off in recessions and reduce their holdings in risky assets. If, in next period, the economy is in an expansionary state consumers are punished twice. First, their financial asset does not pay and second, they miss out on increased dividend payments. Thus, they have to further reduce their position in the real asset to finance consumption. Therefore resulting in a wider distribution of real asset holdings after 500 years.

In Model 1, on the other hand, consumers still trade in the bond but also trade in the additional financial asset that pays off in the good state. If consumers assign a high probability to a recessionary state they will sell the financial asset that pays off in good states and because of the collateral constraint they have to hold the real asset and shorting the bond is limited. In a good state they will lose on the financial asset but gain on the real asset. Hence, the punishment of consumers for a wrong investment is limite. This results in a narrower distribution of asset holdings.

This has also consequences for the distribution of financial wealth. For the baseline economy, the financial wealth share is the narrowest while for Model 2 it is the wides and Model 1 is between the baseline economy and Model 2.

3.3 Volatility

In the next step we look at the equilibrium properties of asset prices in the economy.

Figure 3 shows the mean and volatility of three key variables of interest: Price/Dividend Ratio, Risk Premium, and Risk-Free interest rate. Unsurprisingly, expanding the set of tradable financial assets affects prices in equilibrium as these new financial assets are actively traded by consumers. Furthermore, the Volatility of the Price/Dividend Ratio and Risk Premium increase if new financial assets are introduced while the direction of change for the risk free interest rate depends on the type of financial asset that is introduced, i.e. a financial asset that pays off in expansionary states reduce volatility while financial assets that pay off in recession state increase volaitlity.

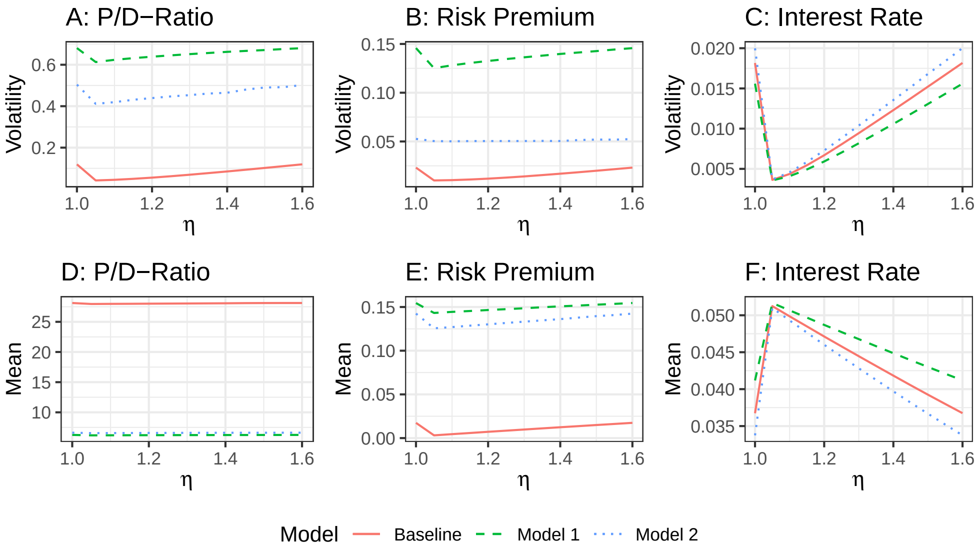

Figure 3: Volatility and Mean of the Price/Dividend Ratio (Panels A and D), the Risk Premium (Panels B and E), and the risk-free interest rate (Panels C and F) in the baseline model and with different financial assets.

Furthermore, expanding the set of tradable financial securities also affects the average interest rat in the economy (Panel F) and in our model the average interest rate increases (Model 1) or decreases (Model 2). This echoes results from static models (see e.g. Elul (1997)) that have shown that additional financial securities have arbitrary effects on the interest rate.

Changes in the risk-free rate are important as they affect the savings behavior. And a lower risk-free rate has two opposing effects. First, a lower interest rate makes bond less attractive compared to real assets and thus consumers will change their portfolio from financial assets to real assets making their portfolio riskier and therefore making the long-run distribution of financial wealth more volatile. The precautionary savings motive works in the opposite direction, i.e. a lower interest rate makes consumers save more to smooth consumption. Which of these two effects dominates depends on the elasticity of

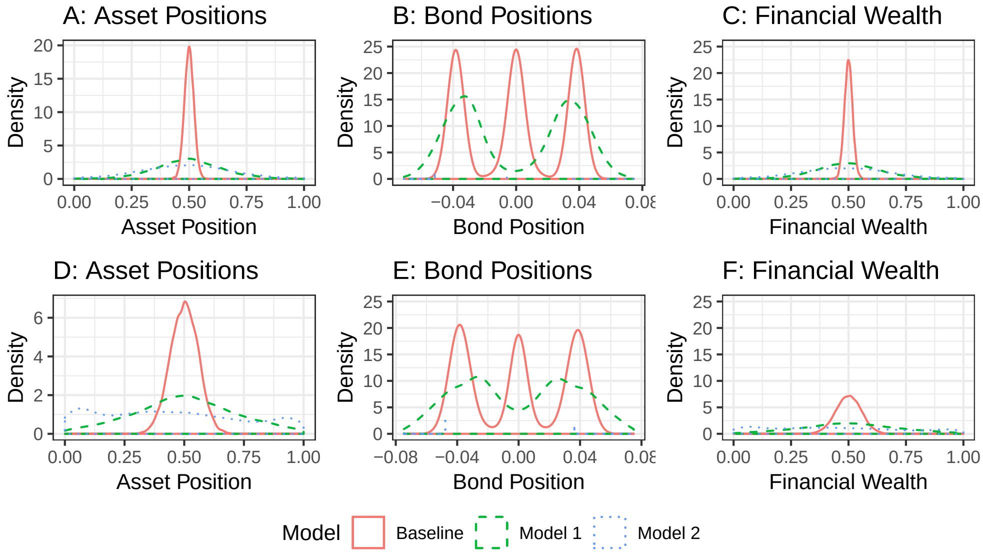

Figure 4: Distribution of Asset and Bond holdings after 500 years. The upper 2 panels show the distribution for $\psi = 2/3$, i.e. the case of CRRA utility and the bottom 2 panels show the distribution for $\psi = 0.35$.

intertemporal substitution.

3.4 The Role of the Elasticity of Intertemporal Substitution

We are now looking at the role of elasticity of intertemporal substitution on equilibrium portfolio choices and asset prices.

Figure 4 shows the bond and asset positions for two different cases of elasticity of intertemporal substitution. In the upper panels the Elasticity of intertemporal substitution is $\psi = 2/3$, i.e. the inverse of the risk aversion and therefore the case of CRRA utility. In the lower panels the Elasticity of Intertemporal substitution is $\psi = 0.35$.

For the case of the baseline model there are no significant changes compared to case of $\psi = 1.5$. For model 2, again the consumers do not hold any

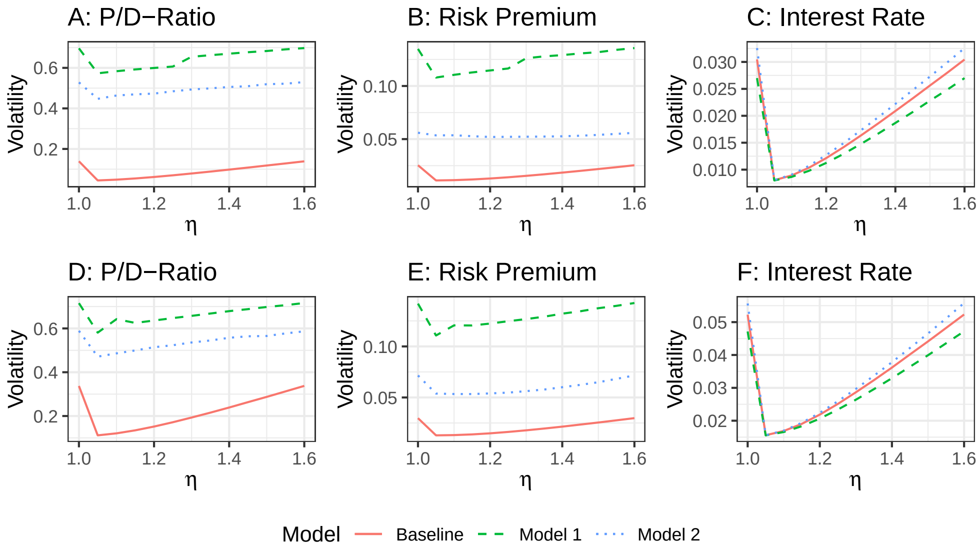

Figure 5: Volatility and Mean of the Price/Dividend Ratio (Panels A and D), the Risk Premium (Panels B and E), and the risk-free interest rate (Panels C and F) in the baseline model and with different financial assets. The upper 3 panels show the results for $\psi = 2/3$, i.e. the case of CRRA-utility. The bottom 3 panels show the results for $\psi = 0.35$.

bonds after 500 years while for model 1 the distribution becomes more even.

The biggest changes are for the positions in the real asset. In particular, the asset positions become more evenly distributed with a lower elasticity of intertemporal substitution. This has also consequences for the distribution of financial wealth in the economy. In particular the distribution of financial wealth becomes more volatile.

These changes in portfolio behavior should also affect equilibrium asset prices. As before, expanding the set of financial securities increases volatility of equilibrium price and the biggest change is in the volatility of the risk-free interest rate which becomes more volatile

4 Conclusion

In this paper we studied the effects of financial innovations on portfolio choices and equilibrium prices in an economy with heterogeneous beliefs and recursive preferences. In our model a financial innovation was interpreted as an additional financial security in addition to a bond and a real asset (Lucas-Tree) that consumers could trade. Our results show that financial innovation increased increased volatility of the Risk-Premium and the Price/Dividend-Ratio, because additional financial securities incentivized consumers to hold riskier portfolios which increases consumption volatility and therefore volatility of equilibrium prices.

The model had a fixed set of tradable financial assets and we compared different economies with different sets of tradable financial assets. However, in the real world the set of financial securities is not fixed and while some financial innovations may become successful products other innovations may disappear and it would be interesting to study such dynamics of financial innovation and its impact on asset prices. However, this would require a different definition of rational beliefs as is currently employed in this paper.

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A Definitions

For the definition of stability and ergodicity use the definitions from Kurz (1994).

Let Ω denote a sample space, $\mathcal{F}$ a σ-field of subsets of Ω, T the shift transformation such that $T(x_t, x_{t+1}, x_{t+2}, ...)$ and Π a probability measure. Define now

$$1_S(x) = \begin{cases} 1 & \text{if x} \in S \\ 0 & \text{if x} \notin S \end{cases} . \tag{32}$$

The relative frequency of the set S visited by the dynamical system given that it start at x as follows

$$m^n(S)(x) = \frac{1}{n} \sum_{k=0}^{n-1} 1_S \left(T^k x \right). \tag{33}$$

Then we define stability and ergodicity as follows

Definition 2 (Stability). *A dynamical system $(\Omega, \mathcal{F}, T\Pi)$ is said to be stochastically stable if for all cylinders $Z \in \mathcal{F}$ the limit of m^n exist Π a.e., and the limit is denoted by*

$$\tilde{m}(S)(x) = \lim_{n \to \infty} m^n(S)(x). \tag{34}$$

Definition 3 (Invariance). *$S \in \mathcal{F}$ is said to be invariant with respect to T if $T^{-1}S = S$. A measurable function is said to be invariant with respect to T if for any $x \in \Omega$, $f(T(x)) = f(x)$.*

Definition 4 (Ergodicity)**.** *A dynamical system is said to be ergodic if* $\Pi(S) = 0$ *or* $\Pi(S) = 1$ *for all invariant sets* S*.*

Definition 5 (Compatibility with the Data)**.** *We say that a probability* $Q \in \mathcal{P}(\Omega)$ *is compatible with the data if*

(a) $(\Omega, \mathcal{F}, Q, T)$ *is stable with a stationary measure* m*. That is, for all cylinders* $S \in \mathcal{F}$

$$m_Q(S) \stackrel{d}{=} \lim_{n\infty} \frac{1}{n} \sum_{k=0}^{n-1} Q(T^{-k}S) = m(S)$$

(b) Q *satisfies the tightness condition* Π*.*

B Derivation of the first order conditions

For ease of notation, we drop the reference to a household h. The maximization problem of the agent can be written as the following Lagrangian:

$$\begin{aligned} \mathcal{L} \;=\; & \left((1-\beta)\left(c_t\right)^{\frac{1-\gamma}{\rho}} + \beta[E_{Q_t}\left[(U_{t+1}g_{t+1})^{1-\gamma} \,| \mathcal{F}_t \right]^{\frac{1}{\rho}} \right)^{\frac{\rho}{1-\gamma}} \\ & -\lambda_t^b \left(c_t + \theta_t q_t + b_t p_t - \theta_{t-1}(q_t + d_t) - \frac{b_{t-1}}{g_t} - e_t \right) - \lambda_t^s \theta_t^h \\ & -\lambda_t^c \left(b_t + (1-m)\theta_t \min_{s^{t+1}|s^t} (q_{t+1} + d_{t+1}) \right). \end{aligned} \tag{35}$$

The lagrange multiplier with respect to the budget constraint is denoted by μ_t^b, for the short-sale constraint μ_t^s and for the collateral constraint μ_t^c. Taking now the derivative with respect to consumption and rearranging yields

$$\frac{\partial \mathcal{L}}{\partial c_t} = (U_t)^{\psi^{-1}} c_t^{-\psi^{-1}} = \lambda_t^b. \tag{36}$$

The derivative with respect to asset purchases is

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \theta_t} &= (U_t)^{\psi^{-1}} \beta E_{Q_t} \left[(U_{t+1} g_{t+1})^{1-\gamma} \right]^{\frac{1-\rho}{\rho}} E_{Q_t^h} \left[(U_{t+1} g_{t+1})^{-\gamma} g_{t+1} \frac{\partial U_{t+1}}{\partial \theta_t} \right] \\ &\quad -\lambda_t^b q_t - \lambda_t^s - \lambda_t^c (1-m) \min_{s^{t+1}|s^t} (q_{t+1} + d_{t+1}), \end{aligned} \tag{37}$$

and because of the envelope theorem the derivative of U_{t+1} with respect to θ_t is given by

$$\frac{\partial U_{t+1}}{\partial \theta_t} = \frac{\partial U_{t+1}}{\partial c_{t+1}} \frac{\partial c_{t+1}}{\partial \theta_t} = (U_{t+1})^{\psi^{-1}} (1-\beta)(c_{t+1}^h)^{-\psi^{-1}} (q_{t+1} + d_{t+1}). \tag{38}$$

Combining the last two equations we get

$$q_t \lambda_t^b = (U_t)^{\psi^{-1}} \beta E_{Q_t} \left[(U_{t+1} g_{t+1})^{1-\gamma} \right]^{\frac{1-\rho}{\rho}} \tag{39}$$

$$E_{Q_t} \left[(U_{t+1})^{\psi^{-1}-\gamma} g_{t+1}^{1-\gamma} (1-\beta) \left(c_{t+1}^h \right)^{-\psi^{-1}} (q_{t+1} + d_{t+1}) \right] \tag{40}$$

$$+\lambda^s + \lambda^c (1-m) \min_{s^{t+1}|s^t} (q_{t+1} + d_{t+1}).$$

The first order conditions for bond holdings can be derived similarly, i.e.

$$p_t \lambda_t^b = (U_t)^{\psi^{-1}} \beta E_{Q_t} \left[(U_{t+1} g_{t+1})^{1-\gamma} \right]^{\frac{1-\rho}{\rho}} \tag{41}$$

$$E_{Q_t} \left[(U_{t+1})^{\psi^{-1}-\gamma} g_{t+1}^{-\gamma} (1-\beta) \left(c_{t+1}^h \right)^{-\psi^{-1}} \right] \tag{42}$$

$$+\lambda^c.$$

C Numerical Algorithm

To solve for the stationary equilibrium we use a time-iteration algorithm. The algorithm proceeds as follows:

Step 0: Set an error-tolerance ϵ and form a grid M over $[0,1]$, Set an initial guess f^0 for policy and price functions.

Step 1: Given a set of policy and price functions f^{n-1}, we obtain a new set of policies and prices f^n by solving the system of equilibrium conditions, the law of motion for the wealth share and optimality conditions (43)-(51) for each gridpoint $(\omega, y) \in M \times \mathcal{Y}$. As the short-sale constraint as well as the margin requirement are not always binding, the Lagrange-Multipliers λ^{hc} and λ^{hs} are not differentiable at edge-cases. Hence, the system of equation is not differentiable. To circumvent the problem we use the Garcia-Zangwill trick (Zangwill and Garcia (1981)) and replace the lagrange multiplier λ^{hs} and λ^{hc} with $\lambda^{hs+} = \max\{0, \lambda^{hs}\}^2$, $\lambda^{hs-} = \max\{0, -\lambda^{hs}\}^2$, $\lambda^{hc+} = \max\{0, \lambda^{hc}\}^2$,

$\lambda^{hc-} = \max\{0, -\mu^{hc}\}^2$. Thus, the system of equations is now as follows:

$$
\begin{aligned}
q_n (U^h)^{\psi^{-1}} (c_n^h)^{-\psi^{-1}} &= (U_n^h)^{\psi^{-1}} \beta E_Q^h \left[\left(U_{n-1}^h(\omega^+, y^+) g(y^+) \right)^{1-\gamma^h} \right]^{\frac{1-\rho^h}{\rho^h}} \qquad (43)\\
& E_Q^h \left[\left(U_{n-1}^h(\omega_{n-1}^+, y^+) \right)^{(\psi^h)^{-1} - \gamma^h} c_{n-1}^h(\omega^+, y^+)^{-\psi^{-1}} \right.\\
& \left. (1-\beta) g(y^+)^{1-\gamma^h} (q_{n-1}(\omega^+, y^+) + d(y^+)) \right]\\
&+ \lambda_n^{hs+} + \lambda_n^{hc+} (1-m) \min_{y^+} (q_{n-1}(\omega^+, y^+) + d(y^+))\\
p_n (U_n^h)^{\psi^{-1}} (c_n^h)^{-\psi^{-1}} &= (U_n^h)^{\psi^{-1}} \beta E_Q^h \left[\left(U_{n-1}^h(\omega^+, y^+) g(y^+) \right)^{1-\gamma^h} \right]^{\frac{1-\rho^h}{\rho^h}} \qquad (44)\\
& E_Q^h \left[\left(U_{n-1}^h(\omega^+, y^+) \right)^{(\psi^h)^{-1} - \gamma^h} c_{n-1}^h(\omega^+, y^+)^{-\psi^{-1}} \right.\\
& \left. (1-\beta) g(y^+)^{-\gamma^h} \right] + \lambda^{hc+}\\
c_n^h &= e^h + \omega^h (q_n + d) - \theta^h q_n - b^h p_n \qquad (45)\\
b_n^1 + b_n^2 &= 0 \qquad (46)\\
\theta^1 + \theta^2 &= 1 \qquad (47)\\
c_n^1 + c_n^2 &= 1 \qquad (48)\\
\omega_n^{h+} &= \frac{\theta^h (q_{n-1}(\omega_n^+, y^+) + d(y^+)) + \frac{b_n^h}{g(y^+)}}{q_{n-1}(\omega^+, y^+) + d(y^+)} \qquad (49)\\
\lambda_n^{hs-} &= \theta_n^h \qquad (50)\\
\lambda_n^{hc-} &= \left(b_n^h + \theta_n^h (1-m) \min_{y^+} (q_{n-1}(\omega^+, y^+) + d(y^+)) \right) \qquad (51)
\end{aligned}
$$

Here, equations (43) and (44) are the first order conditions for asset and bond holdings respectively. Equation (45) is the budget constraint while equations (46)-(48) are the equilibrium conditions, equation (49) is the dynamics for wealthshare and equations (50) and (51) are the modified complementary slackness conditions.

To solve for equilibrium prices, in addition to next periods prices, only next periods consumption and Value-function are needed and not portfolio choices. Thus, we do not need to interpolate next periods portfolio choices.

Step 2: Prices and policy functions are updated until $||f^n - f^{n-1}|| < \epsilon$.

In our application, the grid M has 101 equidistant points and ϵ is set to 10^{-4} and the algorithm is implemented in Matlab. To solve the system of nonlinear equations (43)-(51) one can use a nonlinear-equation solver such as fsolve. For our application, we set ϵ to $1e - 4$ and the number of gridpoints is 101.