A Model of Speculative House Prices

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Abstract

House prices exhibit boom-bust cycles which may not be fully explained by fundamentals. This paper argues that heterogeneous beliefs are one explanation for the observed boom-bust cycles by studying an equilibrium asset prcing model with heterogeneous beliefs in which housing services enter the utility function. We restrict the set of agents' possible beliefs to the subset of rational beliefs in the sense of Kurz (1994). Furthermore, trading on the housing market is subject to transaction costs. Simulation results indicate that disagreement amplifies boom-bust cycles and bigger disagreement leads to larger boom-bust cycles. Transaction costs, on the other hand, have only small effects on the boom-bust cycles.
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1 Introduction

The role of speculation as determinant for asset prices is the subject of vast literature (see e.g. Harrison and Kreps (1978), Basak (2005) or Xiong (2013)). This literature attributes "speculative trading" to differences in beliefs as opposed to other trading motives such as risk sharing due to stochastic income. Real estate is an important asset and for many households a large fraction of wealth is tied up in their own house. Furthermore, real estate markets are also subject to similar boom-bust cycles as stock and bond markets. Yet, as it has been argued by Shiller (2007) and Shiller (2008) these boom-bust cycles are hardly explained by changes in underlying economic fundaments such as population growth or changes in GDP. Additionally, Ely (2013) studies 9 episodes of boom-bust cycles in US real estate markets and argues that speculation has been a driving force of prices in nearly all of them.

The recent episode has spurred some interest in examining the role of expectations in boom-bust cylces of housing prices (see for instance Gelain and Lansing (2014), Kuang (2014) or Granziera and Kozicki (2015)). Additionally, Piazzesi and Schneider (2009) and Burnside, Eichenbaum, and Rebelo (2016) study the role of differences in beliefs in partial equilibrium models featuring a search and matching framework and argue that differences in beliefs may explain housing boom and bust cycles.

In this paper we develop a general equilibrium with heterogeneous agents to study the speculative dynamics of real estate markets. In particular we study an exchange economy in which households get utility from consumption and the service stream provided by housing. Additionally, households can trade real estate and borrow from each other. The key features of the model are (i) agents have heterogeneous expectations, (ii) real estate serves as a collateral for borrowing and (iii) households face transaction costs when trading on the real estate market.

To model heterogeneous beliefs we follow Kurz (1994) and restrict possible beliefs to the subset of rational beliefs. In contrast to rational expectations, Rational Beliefs in the sense of Kurz (1994) has less requirements on the knowledge of the agents. In contrast to Rational Expectations agents do not know the true underlying data generating process but use the empirical distribution of prices and growth rates to form their beliefs. If the agents belief that the data generating process is not stationary then agents' beliefs may not converge to the true process. Rational Beliefs have been successful in explaining excess volatility on financial markets (Kurz and Motolese (2001)) and the equity premium puzzle (Kurz and Beltratti (1997)).

On the other hand, beliefs cannot be arbitrary as agents' beliefs have to satisfy a rationality condition. In particular we say that a belief is rational if the unconditional distribution of agents' belief is the same as the unconditional empirical distribution a condition which is not always satisfied for arbitrary beliefs.[1]

The model emphasizes the role of transaction costs as a determinant for house price volatility. Transaction costs can be direct costs such as notary fees or registration fees[2] but also indirect costs such as search frictions. In real estate markets transaction costs can be a substantial and can be as low as 1.25% for home buyers in Denmark or close to 15% for home buyers in Belgium. The United States with 4.25% transaction costs for home buyers is at the lower end of the spectrum.[3]

Transaction costs make portfolio adjustments after an income shock unattractive. In particular agents experiencing an income shock may decide not to trade to adjust for the income shock. The decision to trade or not to trade depends on the income shock as well as the portfolio the agent is holding, i.e. his current holdings of housing stock and bonds. Hence, as in Grossman and Laroque (1990), potentially creating a 'no-trading zone'. Which in turn affects equilibrium asset prices.

One minor aspect of the model is the collateral constraint households face. If borrowing was not permitted disagreement would still lead to adjustments

[1] One example of arbitrary beliefs are fixed but *wrong* beliefs. Such beliefs would be ruled out by the rational beliefs principle as the unconditional distribution is clearly different from the empirical distribution.

[2] Usually, estimates of transaction costs in real estate markets do not take into account possible tax breaks. Hence, estimates of transaction costs typically provide an upper bound for transaction costs.

[3] Transaction costs typically differ for buyers and seller, however a similar picture emerges when seller are taking into account.

in the agents portfolios, i.e. agents expecting house prices to increase would buy and agents expecting to house prices to decrease would sell. Yet, purchasing housing stock means that agents have to delay consumption into the next period which restricts trading. It is therefore the ability to borrow and build leveraged portfolios that generate the dynamics in our model.

The model is solved numerically using a policy function iteration. The simulation results show that the equilibrium prices in the model have periods in which there are barely any price price changes and periods with large swings in prices. Disagreement amplifies the amplitudes of equilibrium prices, i.e. a larger disagreement implies a larger amplitude. Furthermore, while transaction costs affect equilibrium prices and portfolios their contribution to the size of the amplitude is less than the contribution of beliefs.

The rest of this paper is structured as folows. Section 2 presents the general version of the model. Section 3 we discuss the parameterization for the simulation model. In section 4 we discuss the results for the model without transaction costs and in section 5 the impact of transaction costs is discussed. Section 6 concludes the paper.

2 The Model

2.1 The General Setup

Consider an endowment, an economy with a durable and non-durable consumption good in infinite horizon. We interpret the non-durable consumption good as 'housing services' or 'housing stock'. Time runs from $t = 0$ to ∞. There are I types of consumers:

$$i \in \mathcal{I} = \{1, 2, ..., I\}$$

in the economy. These consumers might differ in many dimensions including their preferences and their endowment of the non-durable consumption good e^i. The consumers might also differ in their initial endowment of housing stock. However, the focus of this paper is on the heterogeneity of beliefs.

There are S possible exogeneous states:

$$s \in \mathcal{S} = \{1, 2, ..., S\}.$$

The state captures both aggregate uncertainty (e.g. aggregate growth) as well as idiosyncratic shocks (e.g. income shocks). The evolution of the economy is captured by the realizations of the shocks over time: $s^t = (s_0, s_1, ..., s_t)$. We assume that the shocks follow a markov-process with the transition probabilities $\pi(s, s')$.

Consumers: First, we make the following assumption about the consumer:

Assumption 1. 1. *Each agent believes the economy is markovian.*

2. *Each agent beliefs that no single agent can affect the equilibrium.*

In each state s^t, each consumer is endowed with some endowment $e_t^i = e^i(s_t)$ units of the non-durable consumption good. The aggregate endowment in the economy is $\bar{e}_t = \sum_{i \in \mathcal{I}} e_t^i$ and the growth rate is denoted by $g_t = \frac{\bar{e}_t}{\bar{e}_{t-1}}$. Furthermore, we assume that they have recursive preferences. With recursive preferences the temporal resolution of uncertainty matters and preferences are not separable over time. In general, recursive preferences take the following form:

$$U_t = F(c_t, CE(U_{t+1})), \tag{1}$$

where $F(\cdot, \cdot)$ is a time aggregator and $CE(\cdot)$ is the certainty equivalent. Here we focus on the form proposed by as in Epstein and Zin (1989) and Weil (1990). In particular, Consumers take the sequence of prices $\{p_t, q_t\}$ as given and maximize the following recursive utility functoion:

$$U_t^i = \left((1-\beta) \left(c_t^i \right)^{\frac{1-\gamma^h}{\rho^i}} + \beta E_{Q_t^i} \left[\left(U_{t+1}^i g_{t+1} \right)^{1-\gamma^i} | \mathcal{F}_t \right]^{\frac{1}{\rho^i}} \right)^{\frac{\rho^i}{1-\gamma^i}}, \tag{2}$$

with β as the subjective discount factor, γ^i as the coefficient of relative risk aversion, and the Elasticity of Intertemporal Substitution $\psi^i \geq 0$. The parameter ρ^i is defined as $\rho^i := (1-\gamma^i)/(1-\frac{1}{\psi^i})$. And Q_t^i represents the subjective (effective) beliefs of agent h subject to the information set $\mathcal{F}_t$.

Consumption c_t^h is a consumption bundle consisting of a non-durable consumption good $\tilde{c}_t^i$ and and housing h_t^i and we use the following CES-aggregator:

$$c_t^h = \left[\alpha(\tilde{c}_t^i)^{\frac{\epsilon-1}{\epsilon}} + (1-\alpha)(h_t^i)^{\frac{\epsilon-1}{\epsilon}}\right] \tag{3}$$

If $\epsilon \to \infty$ the two goods become perfect substitutes and perfect complements if $\epsilon \to 0$. If $\epsilon \to 1$ we have the Cobb-Douglas specification.

Housing: Households can buy housing services at the price $q_t = q(s^t)$. Households do not only receive utility from housing services but the housing endowment can be used as a collateral for borrowing. It is natural to assume that houses cannot be sold short, i.e. $h_t^i \geq 0$. Furthermore, we normalize the total supply of housing to 1.

In our model, trading the real asset is subject to transaction costs. In particular, households pay $\tau q_t|h_t^i - h_{t-1}^i|$ in transaction costs when purchasing or selling the house. We do not assume that transaction costs are a deadweight loss but that they are redistributed as a lump-sum payment after agents made their decisions. The lump-sum payment l_t to an individual investor is:

$$l_t^i = \frac{1}{I}\sum_{i\in\mathcal{I}} \tau q_t|h_t^i - h_{t-1}^i|. \tag{4}$$

Bond: In addition to purchasing real assets, consumers can also borrow subject to collateral constraints. The agents borrow by selling $b_t^i = b^i(s^t)$ units of a one-period bond which pays one unit of the consumption good in the next period at price $p_t = p(s^t)$ and use their holdings of the real asset as collateral. In particular, we consider a collateral constraint of the following form:

$$b_t^i + m h_t^i \min_{s^{t+1}|s^t} q_{t+1} \geq 0. \tag{5}$$

The maximization problem is subject to the intertemporal budget constraint:

$$c^i_t + q_t h^i_t + p_t b^i_t + \tau q_t |h^i_t - h^i_{t-1}| \leq e^i_t + b^i_{t-1} + q_t h^i_{t-1} + l_t, \tag{6}$$

the short-sale constraint on housing

$$h^i_t \geq 0, \tag{7}$$

and the collateral constraint

$$b^i_t + m h^i_t \min_{s^{t+1}|s^t} q_{t+1} \geq 0. \tag{8}$$

2.2 First Order Conditions

Because the transaction cost $\tau q_t |h^i_t - h^i_{t-1}|$ the maximization problem is not differentiable at $h^i_t = h^i_{t-1}$. Because of the non-differentiability the price q_t households may not agree on the price q_t for the housing asset. To circumvent this non-differentiability we transform this into a dual problem[4], following Buss and Dumas (2015). In particular we split up the trades into buying housing stock and selling housing stock and introduce a shadow variable r^i_t.

The shadow variable r^i_t satisfies the following condition:

$$1 - \tau \leq r^i_t \leq 1 + \tau. \tag{9}$$

And we can rewrite the maximization problem as follows

$$\max_{c^i_t, h^i_t, b^i_t} \quad U_t = \left[(1-\beta) \left(c^i_t \right)^{\frac{1-\gamma^i}{\rho^i}} + \beta E_{Q^i_t} \left[\left(U^i_{t+1} g_{t+1} \right)^{(1-\gamma^i)} \right]^{1/\rho^i} \right]^{\frac{\rho^i}{1-\gamma^i}} \tag{10}$$

$$\text{s.t.} \qquad \tilde{c}^i_t + q^{bi}{}_t + (h^i_t - h^i_{t-1}) r^i_t q_t - e^i_t - \frac{b^i_{t-1}}{g_t} = 0 \tag{11}$$

$$b^i_t + m h^i_t \min_{s^{t+1}|s^t} q_{t+1} \geq 0 \tag{12}$$

[4]Details are provided in Appendix B

The description of the optimization problem implies that portfolio choices in period t will depend on choices made in period $t-1$ as well as the current state s_t and the beliefs Q_t:

$$
\begin{aligned}
h_t^i &= h^i(h_{t-1}^i, b_{t-1}^i, p_t, q_t, s_t, Q_t^i) &(13)\\
b_t^i &= b^i(h_{t-1}^i, b_{t-1}^i, p_t, q_t, s_t, Q_t^i) &(14)
\end{aligned}
$$

Therefore, equilibrium allocation and prices in period t will depend on the distribution of assets $(h_{t-1}^i, b_{t-1}^i)_{i\in\mathcal{I}}$ as well as distribution of beliefs.

Ignoring the short-sale constraint[5], the first order condition with respect to h_t^i is

$$
\begin{aligned}
r_t^i q_t = & -\frac{\lambda_t^{ic}}{\lambda^{ib}} \min_{s^{t+1}|s^t} + \frac{1-\beta}{\lambda_t^{ib}} \left(U_t^i\right)^{\frac{\rho^i}{1-\gamma^i}-1} \left(c_t^i\right)^{\frac{1-\gamma^i}{\rho^i}-1} \frac{\partial c_t^i}{\partial h_t^i} \\
& + \beta \left(U_t^i\right)^{\frac{\rho^i}{1-\gamma^i}-1} E_{Q_t^i}\left[\left(U_{t+1}^i g_{t+1}\right)^{1-\gamma^i}\right]^{\frac{1}{\rho^i}-1} \\
& E_{Q_t^i}\left[\left(U_{t+1}^i g_{t+1}\right)^{-\gamma^i} \frac{\lambda_{t+1}^{ib}}{\lambda_t^{ib}} g_{t+1} r_{t+1}^i q_{t+1}\right]
\end{aligned}
$$

With λ_t^{ib} as the lagrange multiplier associated with the budget constraint and λ_t^{ic} the lagrange multiplier associated with the collateral constraint.

The valuation of a house can be split into three parts. The first term is the collateral value of the house, i.e. the value an agent attaches to the house because he can borrow against it. The second term is the marginal value of an additional unit of housing, which in our model is interpreted as imputed rent. The third term is the marginal resale value of the house.

If the collateral constraint constraint is not binding and there are no transaction costs, i.e. $\tau = 0$, then a forward iteration shows that the value of the house is the present value of future imputed rents received from holding the house.

If transaction costs are greater than zero, than the resale value of the house declines. In the case of 100% transaction costs the resale value of the house

[5]In our simulation excercise h^i was always bounded away from 0. Therefore the presence of the short-sale constraint should have no effects on equilibrium prices and allocation.

becomes zero. Yet, this does not imply that increasing transaction costs reduce the value of housing. In particular, if households trade less on the housing market then they might start to borrow more to hedge consumption risk which implies that the collateral value of the house increases. Hence, whether increasing transaction cost increase or decrease prices is not entirely clear.

2.3 Equilibrium

In equilibrium goods and housing markets have to clear, therefore we have the following market clearing conditions:

- goods market clears

$$\sum_{i \in \mathcal{I}} c_t^i = 1 \tag{15}$$

- housing market clears

$$\sum_{i \in \mathcal{I}} h_t^i = 1 \tag{16}$$

- bond market clears

$$\sum_{i \in \mathcal{I}} b_t^i = 0 \tag{17}$$

The market clearing conditions imply now that prices in period t are a function of endogeneous state variables, i.e. portfolio choices in previous period, the current state of the economy s_t and beliefs $\mathcal{Q}_{i \in \mathcal{I}}$:

$$\begin{bmatrix} q_t \\ p_t \end{bmatrix} = \Phi\left(\left(h_{t-1}^i, b_{t-1}^i\right)_{i \in \mathcal{I}}, s_t, \left(Q_t^i\right)_{i \in \mathcal{I}}\right) \tag{18}$$

2.4 State Variables

Due to the transaction costs we have two state variables in the model. In our model it is the financial wealth share of the agents and the holdings of the housing asset of agent 1.

We introduce the financial wealth share. From the budget constraint, we can see that agents enter a period t with wealth W_t^i:

$$W_t^i = h_{t-1}^i r_t^i q_t + \frac{b_{t-1}^i}{g_t} \tag{19}$$

Using the market clearing conditions and aggregate the financial wealth of the agents we get the following aggregate wealth:

$$AW_t = \sum_{i \in \mathcal{I}} W_t^i = \sum_{i \in \mathcal{I}} h_{t-1}^i r_t^i q_t \tag{20}$$

Thus, the financial wealth share of an agent is simply her financial wealth divided by the aggregate wealth:

$$w_t^i = \frac{h_{t-1}^i r_t^i q_t + \frac{b_{t-1}^i}{g_t}}{\sum_{i \in \mathcal{I}} h_{t-1}^i r_t^i q_t} \tag{21}$$

2.5 The Structure of Beliefs

So far, we have taken the beliefs Q_t^i as given and we are now turning to the construction of rational beliefs.

Let X denote the state-space of data and observable $(s_t, p_t, q_t, (h_t^i, b_t^i, c_t^i)_{i \in \mathcal{I}})$ for all t and X^∞ the state space for the entire sequence. The Borel σ field generated by X^∞ will be denoted as $\mathcal{B}(X^\infty)$. The true stochastic process of the economy is described by a stochastic dynamic system $(X^\infty, \mathcal{B}(X^\infty), T, \Pi)$, where T denotes the shift-transformation[6] and Π the true probability measure.

We define now a rational belief:

Definition 1. *(Rational Beliefs) A sequence of effective beliefs $\{Q_t^i\}_{t=0}^\infty$ are a rational belief if the sequence is stable and ergodic and it induces a stationary measure that is compatible with the data and equivalent to the one induced by the empirical measure Π.*

[6] The shift transformation T is defined as $x_{t+1} = Tx_t$. It is not assumed to be invertible, i.e. $T^{-1}x_{t+1} \neq x_t$, which implies that any future evolution is not associated with a unique past.

This definition states that rational beliefs are compatible with the empirical data which makes it impossible to reject a rational belief by examining the data. However, rational beliefs still allows for mistakes as the definition does not require the belief to the true probability. It is important to note the rational beliefs principle rules out fixed (or dogmatic) beliefs, unless they believe that the empirical distribution is the true distribution. Do note that this definition of rational beliefs does not require agents to know the equilibrium map (18), instead agents deduce the relationships between variables on the observable data.

However, the definition of rational beliefs in the sense that it does tell us how we should construct rational beliefs or how agents should learn from the available data. Furthermore, a belief on X^∞ is a rather complicated object and it may prove impossible to check stability. Hence, instead modelling the learning process we pose the problem differently: Given the dynamic system $(X^\infty, \mathcal{B}(X^\infty), T, \Pi)$ we construct a sequence of effective beliefs that are rational beliefs.

To include the beliefs of the agents we follow Kurz and Schneider (1996) expand the probability space and include the sequence of generating variables $(n_t^i)_{t=1}^\infty$. Now, agent h forms a belief Q_t^h on $\left(\left(X \times \mathcal{N}^i\right)^\infty, \mathcal{B}\left((X \times \mathcal{N})^\infty\right)\right)$, where $\mathcal{N}^i := \{0, 1\}$ denotes the state space of n_t^i, and $\mathcal{B}\left(\left(X, \mathcal{N}^i\right)^\infty\right)$ is the Borel σ-field generated by $\left(X \times \mathcal{N}^h\right)^\infty$. Now let $n^{it} := (n_1^i, n_2^i, ..., n_t^i)$, i.e. the history of generating variables n_t^i up to period t. Then, each finite history n^{it} determines agent $i's$ effective belief in period t denoted by $Q_t^i(A) = Q^h(A|n^{it})$ for $A \in \mathcal{B}(X^\infty)$, which is a probability measure on $(X^\infty, \mathcal{B}(\Sigma^\infty))$. The analysis is simplified by the following assumption

Assumption 2. *The marginal distribution for n_t^i with respect to Q_t^i is i.i.d. with $Q^i(n_t^i = 1) = \mu^i$.*

Assumptions 1 and 2 imply that the effective belief Q_t^i is solely determined by the generating variable n_t^i, i.e. $Q_t^i(A) = Q^h(A|n_t^i)$ for $A \in \mathcal{B}(\Sigma^\infty)$. Hence, we interpret the variable n_t^i as describing the state of belief of agent i in period t.

For example, the belief Q^i supports a regime switching model, then n_t^i describe the regime in which agent h believes the economy is. For instance, if $n_t^i = 1$ agent i may be optimistic about the economy while $n_t^i = 0$ corresponds to a pessimistic state of belief.

Furthermore, even though households switches between beliefs are i.i.d. the description above allows us to model a wide range of joint dynamics between beliefs and states. Before we state the Conditional Stability Theorem, we introduce some important notation.

Let Π_k^i denote the conditional probability of $\hat{\Pi}^i$ given a particular sequence of effective beliefs $k \in (\mathcal{N}^i)^\infty$:

$$\hat{\Pi}_k^i(\cdot) : (\mathcal{N}^i)^\infty \times \mathcal{B}(X^\infty) \mapsto [0,1] \tag{22}$$

For each $A \in \mathfrak{B}(X^\infty)$, $\hat{\Pi}_k^i$ is a measurable function of k and for each k, $\hat{\Pi}_k^i(\cdot)$ is a probability on $(N^\infty, \mathcal{B}(X^\infty))$. For $A \in \mathcal{B}(X^\infty)$ and $B \in \mathcal{B}((\mathcal{N}^h)^\infty)$, we have

$$\hat{\Pi}^i(A \times B) = \int_{k \in B} \hat{\Pi}_k^i(A) \bar{\mu}^i(dk). \tag{23}$$

Also, as we noted above,

$$\begin{aligned} \Pi(A) &= \hat{\Pi}^i(A \times (\mathcal{N}^i)^\infty), \forall A \in \mathcal{B}(X^\infty) \\ \bar{\mu}^i(B) &= \hat{\Pi}(X^\infty \times B), \forall B \in \mathcal{B}((\mathcal{N}^i)^\infty). \end{aligned} \tag{24}$$

If $(\Omega^i, \mathfrak{B}^i, T, \hat{\Pi}^i)$ is a stable dynamical system with a stationary measure $m^{\hat{\Pi}^i}$, we define the two marginal measures of $m^{\hat{\Pi}^i}$ as follows:

$$m(A) := m^{\hat{\Pi}^i}(A \times (\mathcal{N}^i)^\infty), \forall A \in \mathcal{B}(X^\infty) \tag{26}$$

$$m_{Q^i}(B) := m^{\hat{\Pi}}(X^\infty \times B), \forall B \in \mathcal{B}((\mathcal{N}^i)^\infty). \tag{27}$$

Also, let $\hat{m}_k$ denote the stationary measure of $\hat{\Pi}_k^i$, which is a measure on $(X^\infty, \mathfrak{B}(X^\infty))$. Given the construction of the dynamical system, we have the following theorem:

Theorem 1. *(Conditional Stability Theorem, Kurz and Schneider (1996)).*
Let $(\Omega^i, \mathcal{B}^i, T, \hat{\Pi}^i)$ be a stable and ergodic dynamical system. Then,

1. $(X^\infty, \mathcal{B}^\infty, T, \hat{\Pi}^i_k)$ *is stable and ergodic for* $\hat{\Pi}^i$ *a.a. k.*

2. $\hat{m}^i_k$ *is independent of k,* $m^i_k = m = \Pi$.

3. *If* $(X^\infty, \mathfrak{B}(X^\infty), T, \hat{\Pi}^i_k)$ *is stationary, then the stationary measure of* $\hat{\Pi}^i_k$ *is* Π. *That is*

$$\hat{m}^i_k = m = \Pi.$$

So far, our discussion on constructing rational beliefs did assume that agents do not know the equilibrium map 18. However, to simplify the computational model we assume that agents do know the equilibrium map 18. This implies that once agents have chosen their portfolios next periods prices depend only on the state s_t and the distribution of belief $(Q^i_t)_{i \in \mathcal{I}}$. This simplifies the construction of a computational model as agents need to form beliefs only over the exogeneous variables, i.e. they form beliefs over $\left(s_t, (n^i_t)_{h \in \mathcal{H}}\right)$

Example. *To illustrate the ideas, we consider an example similar to our simulation model discussed in a later section. Consider an economy with two exogeneous states (e.g. high growth and low growth) and two agents. Both agents can be either optimistic in the sense that she assigns a higher probability on high growth then empirically observed or pessimistic in the sense that she assigns a lower probability on high growth than empirically observed. Now, our state-space consists of 8 states. In particular, we have the tuple* $\{g_t, n^1_t, n^2_t\}$. *Now, in period t agents form beliefs not only over dividends but also over the distribution of future generating variables* $\{n^1_{t+1}, n^2_{t+1}\}$.

This implies that the sequence of effective beliefs of a household Q^i_t *must have the same stationary distribution as the tuple* $\{g_t, n^1_t, n^2_t\}^\infty_{t=0}$. *If the tuple has a Markov transition matrix* Γ *and the beliefs are represented by two transition matrices* F^h_H, F^h_L *the rationality condition implies that*

$$\mu^i F^i_H + (1 - \mu^i) F^i_L = \Gamma. \tag{28}$$

□

Now, using the generating variable, we can rewrite the portfolio choice

(13)-(14) in terms of generating variables rather than beliefs Q_t^h:

$$h_t^i = h^i\left(\theta_{t-1}^i, b_{t-1}^i, p_t, q_t, s_t, n_t^i\right) \tag{29}$$

$$b_t^i = \theta^i\left(\theta_{t-1}^i, b_{t-1}^i, p_t, q_t, s_t, n_t^i\right). \tag{30}$$

We define the stochastic primitives y_t as follows:

$$y_t = \left(s_t, (n_t^i)_{i \in \mathcal{I}}\right) \quad \forall t. \tag{31}$$

The state space of the stochastic primitives is now $\mathcal{Y}$. We assume that the $\{y_t\}_{t=0}^{\infty}$ is a stable Markov process with a time homogeneous transition probability $P : \mathcal{Y} \rightarrow \mathcal{P}(\mathcal{Y})$, where $\mathcal{P}(\mathcal{Y})$ denotes the space of probability measures on $\mathcal{Y}$.

3 The Simulation Model

We assume that there are only 2 agents in the economy, that is, $I = 2$. We also assume that there are two growth states, i.e. $g_t \in \{\bar{g}, \underline{g}\}$. The empirical distribution $\{g_t\}$ follows a markov-process:

$$\Psi = \begin{bmatrix} \phi & 1-\phi \\ 1-\phi & \phi \end{bmatrix}. \tag{32}$$

Then, the stationary transition probability matrix has to satisfy the following conditions:

- the empirical distribution for the process g_t is specified by transition probability matrix Ψ.
- the marginal distribution for n_t^i is i.i.d with frequency of $\{n_t^i = 1\} = \alpha^i$.

Here, we use a specification similar to Kurz and Motolese (2001) as we know that the beliefs are compatible with the stationary distribution and it can generate large fluctuations. Furthermore, this specification allows for correlation between the three variables (g_t, n_t^1, n_t^2). We assume that the 8×8 matrix Γ has

Table 1: Transition probabilities of the states of beliefs.

State Next Period → Current Period ↓	Optimistic	Disagreement	Pessimistic
Optimistic	0.50	0.36	0.14
Disagreement	0.14	0	0.86
Pessimistic	0.14	0	0.86

the following structure:

$$\Gamma = \begin{bmatrix} \phi A & (1-\phi)A \\ (1-\phi)A & \phi A \end{bmatrix} \tag{33}$$

A is a 4×4 matrice defined by 6 parameters (α^1, α^2, a) and $a = (a^1, a^2, a^3, a^4)$ as follows:

$$A = \begin{bmatrix} a^1 & \alpha^1 - a^1 & \alpha^2 - a^1 & 1 + a^1 - \alpha^1 - \alpha^2 \\ a^2 & \alpha^1 - a^2 & \alpha^2 - a^2 & 1 + a^2 - \alpha^1 - \alpha^2 \\ a^3 & \alpha^1 - a^3 & \alpha^2 - a^3 & 1 + a^3 - \alpha^1 - \alpha^2 \\ a^4 & \alpha^1 - a^4 & \alpha^2 - a^4 & 1 + a^4 - \alpha^1 - \alpha^2 \end{bmatrix} \tag{34}$$

We also have to specify the transition probability matrices that represent the beliefs of the agents. As noted above, agent $h \in \{1, 2\}$ in period t uses F_1^i when his generating variable is $n_t^1 = 1$ and F_2^i when his generating variable is $n_t^1 = 0$. The rationality of belief condition implies that

$$\alpha^i F_1^i + (1 - \alpha^i) F_2^i = \Gamma. \tag{35}$$

Thus to fully pin down a traders' belief we only have to specify F_1^i while F_2^i can be inferred from Γ and F_1^i. The matrix F_1^i is parametrized by η^i as follows:

$$F_1^i(\eta^h) = \begin{bmatrix} \phi \eta^h A & (1 - \eta^i \phi)A \\ (1-\phi)\eta^i A & (1 - (1-\phi)\eta^i)A \end{bmatrix} \tag{36}$$

From the above equation one can see that if $\eta^i > 1$ a trader places more weight on the growth states, i.e. he is overly optimistic that the economy grows when his beliefs are given by F_1^i. Furthermore, the larger the η^i implies a more optimistic trader. Furthermore, parameter α^i determines the frequency of optimistic beliefs, when $\alpha^i = 0.5$ then optimistic and pessimistic have the same frequency while $\alpha^i > 0.5$ implies that a trader is more often optimistic then pessimistic. This has also implications for pessimistic beliefs. In particular if $\eta^i > 1$ and $\alpha^i > 0.5$ then beliefs are more asymmetrically distributed to satisfy the rationality condition.

For the beliefs of the agents we follow Kurz and Motolese (2001) and set $(a_1, a_2, a_3, a_4) = (0.5, 0.14, 0.14, 0.14)$. Table 1 provides an interpretation of the parameters $a = (a_1, a_2, a_3, a_4)$. One can see that beliefs are correlated which reflects some communication across agents (see Nakata (2007) for a discussion)[7]. Furthermore, we assume that $\alpha^1 = \alpha^2 = \alpha = 0.57$. The maximum value for η is $1/0.57 \approx 1.7$ and we will examine several different cases of η. In particular we consider $\eta^1 = \eta^2 = \eta =\in \{1.2, 1.4, 1.6\}$.

Following Mehra and Prescott (1985) we consider the following transition probability matrix for Ψ:

$$\Psi = \begin{bmatrix} 0.43 & 0.57 \\ 0.57 & 0.43 \end{bmatrix}, \tag{37}$$

and set $\overline{g} = 1.054$ and $\underline{g} = 0.982$.

Our choices for preferences follow the literature. We set the time-preference parameter to $\beta = 0.96$, the coefficient of relative risk-aversion is set to $\gamma = 1.5$ which is standard in the literature. On the other hand, for the value of the EIS there is a bigger range of estimates. Some authors estimate a rather low value for the EIS, for example Hall (1988) estimates a value much smaller than 1, while several asset pricing models (e.g., Collin-Dufresne, Johannes, and Lochstoer (2014) or Bansal and Yaron (2004)) have used a EIS greater than 1. An EIS greater than 1 is needed to capture the negative correlation between consumption volatility and the price/dividend-ratio. For the baseline model we set the elasticity of intertemporal substitution for both agents to $\psi = 1.5$, a

[7] If we would follow Nakata (2007) and add communication to our model the exogeneous state-space would increase to 32 variables which drastically increases computational burden, hence we use correlation of beliefs as a short-cut for communication.

Table 2: Model Parameters

Variable	Meaning	Value
γ	Risk Aversion	2.0
ψ	EIS	1.5
η	Beliefs	1.2,1.4,1.6
α		0.95
ϵ	Elasticity of Substitution	1.25
τ	Transaction Costs	$0\% - 0.20\%$
m	Collateral Requirements	0.5

value which is in line with the asset pricing literature.

For the elasticity of substitution between housing good and consumption good, we set $\epsilon = 1.25$ as in Piazzesi, Schneider, and Tuzel (2007).

Furthemore, we set the parameter m to 0.5, which implies a Loan-to-Value ration of 50%.

We summarize the important parameters for the simulation study in table 2

The model is solved using a policy function iteration and the details of the solution algorithm are outlined in appendix C The simulation results are obtained using one sample path over 10,000 periods.

4 Results for the Model without Transaction Costs

We are now presenting the results of our simulation exercises. To gain some intuition, we start with the model without transaction costs. We then move on to the model with transaction costs.

We first look at the model without transaction costs, i.e. $\tau = 0$. Remember that from the first order conditions we had the imputed rent defined as:

$$\frac{(1-\beta)}{\lambda^{ib}} (U_t^i)^{\frac{\rho^i}{1-\gamma^i}-1} (c_t^i)^{\frac{1-\gamma^i}{\rho^i}} \frac{\partial c_t^i}{\partial h_t^i}. \tag{38}$$

	Model Simulations		
	$\eta^h = 1.2$	$\eta^h = 1.4$	$\eta^h = 1.6$
Trading Volume	0.02	0.04	0.05
Price/Rent Ratio			
Mean	8.84	10.41	12.05
Std. Dev	1.55	3.33	5.12
House Prices			
Mean	1.58	1.69	1.81
Std. dev	0.45	0.94	1.5
imputed rent			
Mean	0.05	0.06	0.06
Std. dev	0.01	0.02	0.03

Table 3: Simulation Results for the model without transaction costs

Therefore, we can define the price/rent ratio in period t in the model as

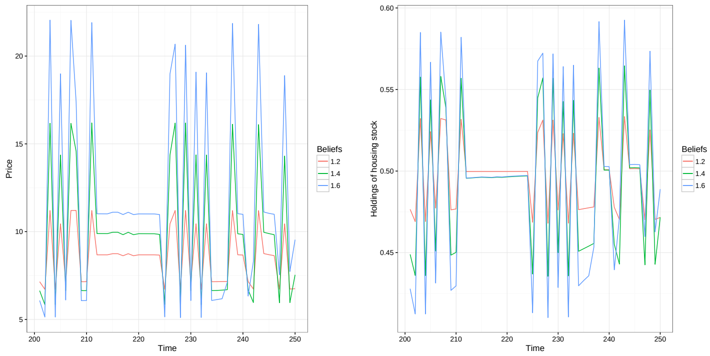
$$PR_t = \frac{q_t}{rent_t} \tag{39}$$

Table 3 shows the simulation results for the model without transaction costs. In particular we look at 4 variables: Trading Volume, House Price, Price/Rent-Ratio and Rent. For the latter 3 we look at the mean and standard deviation. The trading volume is defined as follows:

$$\frac{1}{T}\sum_{t=2}^{T} |h_t^1 - h_{t-1}^1| \tag{40}$$

Unsurprisingly, trading volume increases if disagreements become bigger. Furthermore, the mean and standard deviation of the other 3 variable also increase with disagreements.

In particular, one can see that the change in mean and standard deviation for the price/rent ratio incerases stronger with the disagreement than imputed rent. The mean of the price/rent ratio increases because the collateral value of the house increases. In particular with bigger disagreement, agents have a stronger incentive to trade housing stock, i.e. if one agent has optimistic



(a) Prices (b) Housing stock

Figure 1: This figure shows sample paths for prices and holdings of housing stock in an economy without transaction costs. The beliefs are set to $\eta^h = 1.2$, $\eta^h = 1.4$ and $\eta^h = 1.6$.

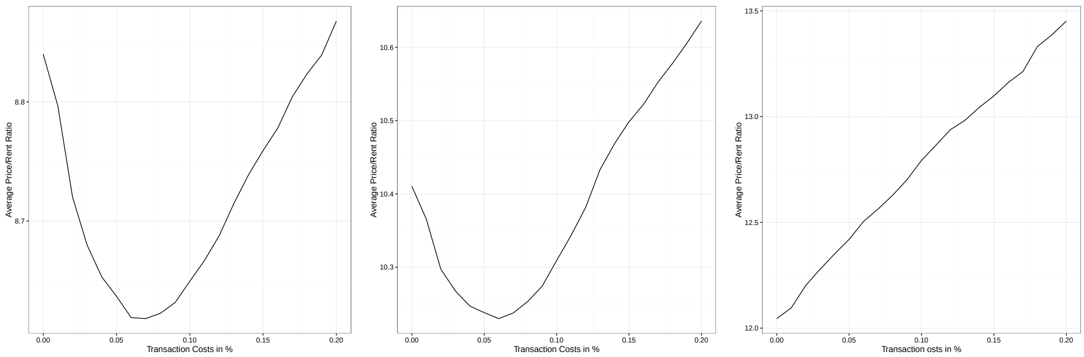
expectations he will start to buy. In order to buy housing stock he will have to borrow. Now, with a greater η^h this means a stronger incentive to buy when optimistic, hence households have to borrow more. Hence, increasing the collateral value of the house.

The collateral constraint also plays an important role in amplifying the effects of disagreement, i.e. with a larger disagreement the fire sale dynamics for the house prices become more severe. Thus increasing volatility of the price/rent ratio.

To further understand the results, we look at sample paths for the price/rent ratio and the holdings of housing stock. These sample paths are shown in Figure 1 and show the price/rent ratio and the holdings of housing stocks.

First, as expected a bigger disagreement leads to a larger amplitude for the price/rent ratio and holdings of housing stock,

Additionally, to periods with large swings in the price/rent ratio there are also tranquil periods in which the price/rent ratio barely changes. These periods coincide with periods in which households barely change their holdings of housing stock.



(a) $\eta^h = 1.2$ (b) $\eta^h = 1.4$ (c) $\eta^h = 1.6$

Figure 2: This graph shows the average price/rent ratio of the simulation model. Transaction costs vary between 0% and 0.20%. The parameter for the beliefs are $\eta^h = 1.2, 1.4, 1.6$

5 The impact of Transaction Costs

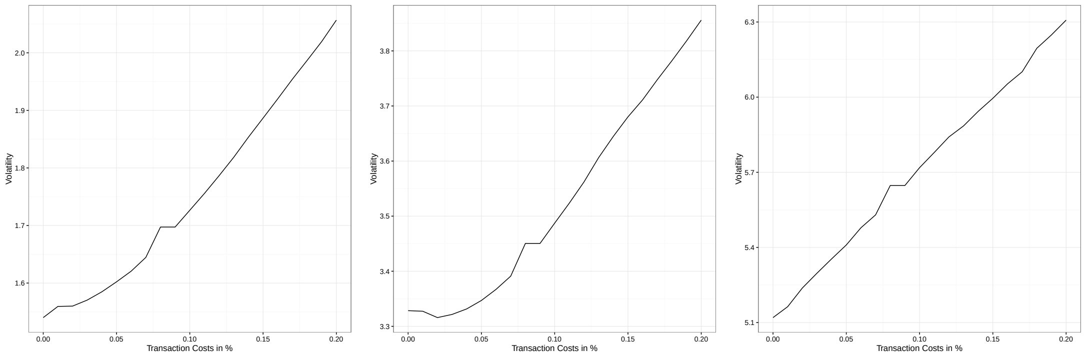
So far, our discussion focussed on the effects of beliefs on house prices. Now, we are turning to the implications of transaction for house prices.

5.1 Equilibrium Asset Prices

Figure 2 shows the how transaction costs affect average price/rent ratio for $\eta = 1.2, 1.4, 1.6$. In all three cases a transaction costs of 0.2% implies a larger mean price. However, in the cases $\eta = 1.2$ and $\eta = 1.4$ the relationship between transaction costs and average house prices not monotone, i.e. the average price first decreases with transaction costs and then increases, because in these two cases the discounted future house price decreases faster than the collateral value increases. While for $\eta = 1.6$ the collateral value increases faster than the discounted future price decreases.

Figure 3 shows the impact of transaction costs on the volatility of normalized house prices. One can see that a transaction cost of about 0.2% increase the volatility of normalized house prices by about 10%. Thus, moderate transaction costs have a much smaller impact than beliefs.

Figure 4 shows a sample path of equilibrium prices. In particular we com-



(a) $\eta^h = 1.20$ (b) $\eta^h = 1.40$ (c) $\eta^h = 1.60$

Figure 3: This graph shows the volatility of the normalized house prices as a function of transaction costs. The parameters for disagreement are $\eta^h = 1.2, 1.4$ and 1.6. Transaction costs vary between 0 and 0.2%

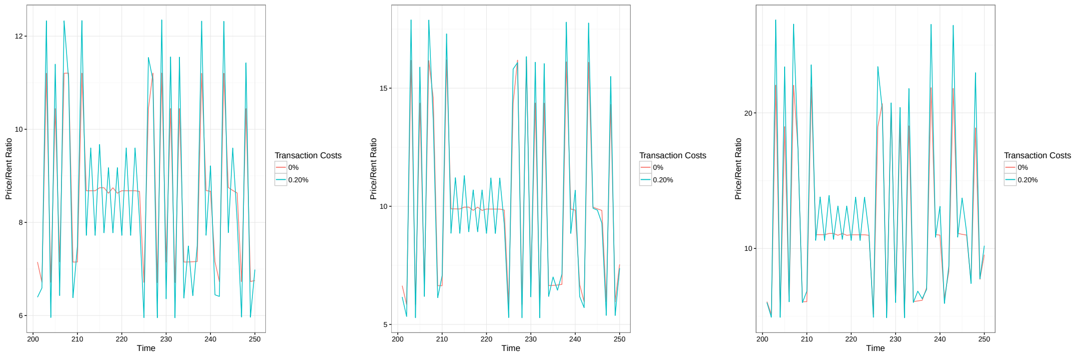
pare the price path with 0.2% transaction to the case of no transaction costs. In all three cases there are no tranquil periods, i.e. those periods which had no price changes for the case without transaction cost do now exhibit some price changes. However, the amplitude is still small compared to previously non-tranquil periods. Additionally, transaction costs also increase the amplited of the price/rent ratio in non-tranquil periods. However, the effect is small compared to the effect of beliefs.

5.2 Trading Volume

We are now turning to the discussion of the effects of transaction costs on equilibrium trading volume of the housing market.

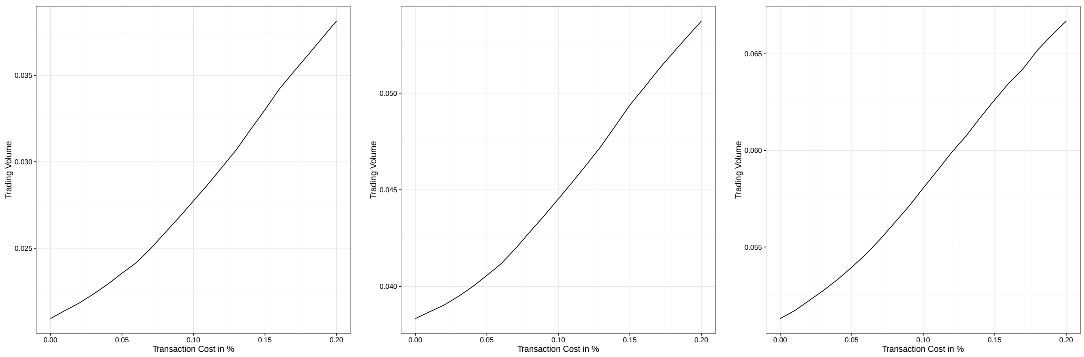
Figure 5 shows the trading volume as a fucnction of transaction costs and beliefs. We can see that trading volume is *increasing* in transaction costs and not decreasing as expected ex-ante. This indicates that for the range of transaction costs considered here, a no-trading zone does not exist.

The non-existence of 'No-trading zones' can also be seen in the sample paths in figure 6. In general transaction costs seem to increase trading, i.e. on top of the trading voume caused by differences in beliefs we also have additional trading from the transaction costs. However, the additional trading



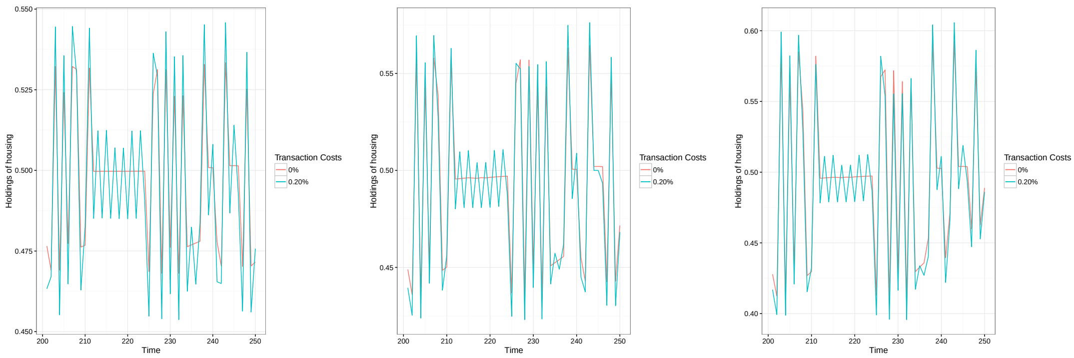
(a) $\eta^h = 1.20$ (b) $\eta^h = 1.40$ (c) $\eta^h = 1.60$

Figure 4: Sample Paths of the Price/Rent Raio for economies with and without transaction costs and $\eta^h = 1.2, 1.4, 1.6$.



(a) $\eta^h = 1.20$ (b) $\eta^h = 1.40$ (c) $\eta^h = 1.60$

Figure 5: Trading Volume as a function of transaction costs and $\eta = 1.2, 1.4, 1.6$.



(a) $\eta^h = 1.20$ (b) $\eta^h = 1.40$ (c) $\eta^h = 1.60$

Figure 6: Sample path of trading on the housing stock for economies with and whithout transaction costs and $\eta = 1.2, 1.4, 1.6$.

generated by transaction costs is smaller than trading generated by differences in beliefs. Furthermore, even in zones where no trading occured without transaction costs we do have some trading.

6 Conclusion

This paper studied the role of heterogeneous beliefs and transaction costs for the dynamics of house prices. Beliefs of agents were restricted to the smaller set of rational beliefs. The simulation results indicate while both, i.e. heterogeneity in beliefs and transaction costs amplify volatility in the economy, heterogeneity in beliefs have a larger effect on volatility than transaction costs.

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A Definitions

For the definition of stability and ergodicity use the definitions from Kurz (1994).

Let Ω denote a sample space, $\mathcal{F}$ a σ-field of subsets of Ω, T the shift transformation such that $T(x_t, x_{t+1}, x_{t+2}, ...)$ and Π a probability measure. Define now

$$1_S(x) = \begin{cases} 1 & \text{if x} \in S \\ 0 & \text{if x} \notin S \end{cases}. \tag{41}$$

The relative frequency of the set S visited by the dynamical system given that it start at x as follows

$$m^n(S)(x) = \frac{1}{n} \sum_{k=0}^{n-1} 1_S\left(T^k x\right). \tag{42}$$

Then we define stability and ergodicity as follows

Definition 2 (Stability). *A dynamical system $(\Omega, \mathcal{F}, T\Pi)$ is said to be stochastically stable if for all cylinders $Z \in \mathcal{F}$ the limit of m^n exist Π a.e., and the limit is denoted by*

$$\tilde{m}(S)(x) = \lim_{n \to \infty} m^n(S)(x). \tag{43}$$

Definition 3 (Invariance). *$S \in \mathcal{F}$ is said to be invariant with respect to T if $T^{-1}S = S$. A measurable function is said to be invariant with respect to T if for any $x \in \Omega$, $f(T(x)) = f(x)$.*

Definition 4 (Ergodicity). *A dynamical system is said to be ergodic if $\Pi(S) = 0$ or $\Pi(S) = 1$ for all invariant sets S.*

Definition 5 (Compatibility with the Data). *We say that a probability $Q \in \mathcal{P}(\Omega)$ is compatible with the data if*

(a) $(\Omega, \mathcal{F}, Q, T)$ is stable with a stationary measure m. That is, for all cylinders

$S \in \mathcal{F}$

$$m_Q(S) \stackrel{d}{=} \lim_{n\infty} \frac{1}{n} \sum_{k=0}^{n-1} Q(T^{-k}S) = m(S)$$

(b) Q satisfies the tightness condition Π.

B Derivation of the first order conditions

The maximization problem of agent i can be written as follows:

$$\begin{aligned} \mathcal{L} &= \max_{c_t^i, h_t^i, b_t^i} \left[(1-\beta) \left(c_t^i \right)^{\frac{1-\gamma^i}{\rho^i}} + \beta E_{Q_t^i} \left[\left(U_{t+1}^i g_{t+1} \right)^{1-\gamma^i} \right] \right]^{\frac{\rho^i}{1-\gamma^i}} & (44) \\ &- \lambda_t^b \left[\tilde{c}_t^i + q_t^b b_t^i + q_t h_t^i + \tau q_t |h_t^i - h_{t-1}^i| - e_t^i - \frac{b_{t-1}^i}{g_t} - q_t h_{t-1}^i - l_t \right] & (45) \\ &- \lambda_t^c \left[b_t^i + m h_t^i \min_{s^{t+1}|s^t} q_{t+1} \right] & (46) \end{aligned}$$

We now split up the trade in housing stock bought and housing stock sold, in particular we have $\hat{h}_t^i \geq 0$ as housing stock bought and $\hat{\hat{h}}_t^i \geq 0$ as housing stock sold. Thus, we have

$$\begin{aligned} h_t^i &= h_{t-1}^i - \hat{\hat{h}}_t^i + \hat{h}_t^i & (47) \\ \hat{h}_t^i \geq 0 & \qquad \hat{\hat{h}}_t^i \geq 0. & (48) \end{aligned}$$

We can therefore rewrite the budget constraint as follows

$$\tilde{c}_t^i + q_t^b b_t^i + q_t h_t^i + \tau q_t \hat{h}_t^i + \tau q_t \hat{\hat{h}}_t^i \leq e_t^i + q_t h_{t-1}^i + \frac{b_{t-1}^i}{g_t} \qquad (49)$$

We can therefore rewrite the maximization problem as follows.

$$\begin{aligned}
\mathcal{L} &= \max_{c_t^i, h_t^i, b_t^i} \left[(1-\beta) \left(c_t^i \right)^{\frac{1-\gamma^i}{\rho^i}} + \beta E_{Q_t^i} \left[\left(U_{t+1}^i g_{t+1} \right)^{1-\gamma^i} \right] \right]^{\frac{\rho^i}{1-\gamma^i}} && (50) \\
&- \lambda_t^b \left[\tilde{c}_t^i + q_t^b b_t^i + q_t h_t^i + \tau q_t \hat{h}_t^i + \tau q_t \hat{\hat{h}}_t^i - e_t^i - \frac{b_{t-1}^i}{g_t} - q_t h_{t-1}^i - l_t \right] && (51) \\
&- \lambda_t^c \left[b_t^i + m h_t^i \min_{s^{t+1}|s^t} q_{t+1} \right] && (52) \\
& -\lambda_t^{ib} \hat{h}_t^i - \lambda_t^{is} \hat{\hat{h}}_t^i && (53)
\end{aligned}$$

Differentiating with respect to $\hat{h}_t^i$ results in

$$(U_t)^{\frac{\rho^i}{1-\gamma^i}-1} \beta E_{Q_t^i} \left[\left(U_{t+1}^i g_{t+1} \right)^{1-\gamma^i} \right]^{\frac{1}{\rho^i}-1} E_{Q_t^i} \left[\left(U_{t+1}^i g_t+1 \right)^{-\gamma^i} g_{t+1} \frac{\partial U_{t+1}^i}{\partial h_t^i} \right] = \lambda_t^b q_t (1+\tau) + \lambda_t^{hb}. \quad (54)$$

While differentiating with respect to$\hat{\hat{h}}_t^i$ yields

$$(U_t)^{\frac{\rho^i}{1-\gamma^i}-1} \beta E_{Q_t^i} \left[\left(U_{t+1}^i g_{t+1} \right)^{1-\gamma^i} \right]^{\frac{1}{\rho^i}-1} E_{Q_t^i} \left[\left(U_{t+1}^i g_t+1 \right)^{-\gamma^i} g_{t+1} \frac{\partial U_{t+1}^i}{\partial h_t^i} \right] = \lambda_t^b q_t (1-\tau) + \lambda_t^{hb}. \quad (55)$$

The last two equations imply that

$$\lambda_t^b (1-\tau) q_t - \lambda_t^{hs} = \lambda_t^b (1+\tau) \lambda_t^{hb} \quad (56)$$

We now replace the two multiplies λ_t^{hb} and λ_t^{hs} with one multiplier r_t defined as follows:

$$\lambda_t^b r_t^i := \lambda_t^b (1+\tau) - \lambda_t^{hb} = \lambda_t^b (1-\tau) - \lambda_t^{hs}. \quad (57)$$

This implies now

$$\begin{aligned} \lambda_t^{hb} &= \lambda_t^b q_t(-r_t^i + 1 + \tau), & (58)\\ \lambda_t^{hs} &= \lambda_t^b q_t(r_t^i - 1 + \tau). & (59) \end{aligned}$$

We rewrite the complementary slackness conditions as follows:

$$\begin{aligned} (-r_t^i + 1 + \tau)\hat{h}_t^i &= 0 & (60)\\ (r_t^i - 1 + \tau)\hat{\hat{h}}_t^i &= 0. & (61) \end{aligned}$$

We also know that

$$1 - \tau \leq r_t^i \leq 1 + \tau. \quad (62)$$

Hence, the budget constraint becomes

$$\tilde{c}_t^i + q_t^b b_t^i + \left(h_t^i - h_{t-1}^i\right) q_t r_t^i \leq e_t^i + \frac{b_{t-1}^i}{g_t} + l_t, \quad (63)$$

and the maximization problem becomes

$$\begin{aligned} \mathcal{L} &= \max_{\tilde{c}_t^i, h_t^i, b_t^i, \hat{h}_t^i \hat{\hat{h}}_t^i} \left[(1-\beta) \left(c_t^i\right)^{\frac{1-\gamma^i}{\rho^i}} + \beta E_{Q_t^i} \left[\left(U_{t+1}^i g_{t+1} \right)^{1-\gamma^i} \right]^{\frac{1}{\rho^i}} \right]^{\frac{\rho^i}{1-\gamma^i}} & (64)\\ &- \lambda_t^b \left[\tilde{c}_t^i + q_t^b b_t^i + (h_t^i - h_{t-1}^i) q_t r_t^i - e_t^i - \frac{b_{t-1}^i}{g_t} \right] & (65)\\ &- \lambda_t^c \left[b_t^i + m h_t^i \min_{s^{t+1}|s^t} q_{t+1} \right] & (66) \end{aligned}$$

By differentiating with respect to $\tilde{c}_t^i$ we get:

$$\left(U_t^i\right)^{\psi^{-1}} \left(c_t^i\right) \frac{\partial c_t^i}{\partial \tilde{c}_t^i} - \lambda_t^b = 0. \quad (67)$$

Now, we differentiate with respect to h_t^i and have

$$0 = -\lambda_t^c m \min_{s^{t+1}|s^t} q_{t+1} - \lambda_t^b r_t q_t + \left(U_t^i\right)^{\frac{\rho^i}{1-\gamma^i}-1}(1-\beta)\left(c_t^i\right)^{\frac{1-\gamma^i}{\rho^i}-1}\frac{\partial c_t^i}{\partial h_t^i} \quad (68)$$

$$+ \left(U_t^i\right)^{\frac{\rho^i}{1-\gamma^i}-1}\beta E_{Q_t^i}\left[\left(U_{t+1}^i g_{t+1}\right)^{1-\gamma^i}\right]^{\frac{1}{\rho^i}-1} E_{Q_t^i}\left[\left(U_{t+1}^i g_{t+1}\right)^{-\gamma^i} g_{t+1}\frac{\partial U_{t1}^i}{\partial h_t^i}\right] \quad (69)$$

Applying the envelope thorem on $\frac{\partial U_{t+1}^i}{\partial h_t^i}$ we get

$$\frac{\partial U_{t+1}^i}{\partial h_{t+1}^i} = \frac{\partial U_{t+1}^i}{\partial c_{t+1}^i}\frac{\partial c_{t+1}^i}{\partial \tilde{c}_{t+1}^i}\frac{\partial \tilde{c}_{t+1}^i}{\partial h_t^i} = \lambda_{t+1}^b r_{t+1} q_{t+1} \quad (70)$$

Thus, the first order condition for h_t^i is now

$$0 = -\lambda_t^c m \min_{s^{t+1}|s^t} q_{t+1} - \lambda_t^b r_t q_t + \left(U_t^i\right)^{\frac{\rho^i}{1-\gamma^i}-1}(1-\beta)\left(c_t^i\right)^{\frac{1-\gamma^i}{\rho^i}-1}\frac{\partial c_t^i}{\partial h_t^i} \quad (71)$$

$$+ \left(U_t^i\right)^{\frac{\rho^i}{1-\gamma^i}-1}\beta E_{Q_t^i}\left[\left(U_{t+1}^i g_{t+1}\right)^{1-\gamma^i}\right]^{\frac{1}{\rho^i}-1} E_{Q_t^i}\left[\left(U_{t+1}^i g_{t+1}\right)^{-\gamma^i} g_{t+1}\lambda_{t+1}^b r_{t+1} q_{t+1}\right] \quad (72)$$

Similarly. the first order conditions for the bond holdings are

$$0 = -\lambda_t^c - \lambda_t^b q_t^b + \left(U_t^i\right)^{\frac{\rho^i}{1-\gamma^i}-1}\beta E_{Q_t^i}\left[\left(U_{t+1}^i g_{t+1}\right)^{1-\gamma^i}\right] \quad (73)$$

$$E_{Q_t^i}\left[left(U_{t+1}^i g_{t+1}\right)^{-\gamma^i}\lambda_{t+1}^b\frac{\partial c_{t+1}^i}{\partial \tilde{c}_{t+1}^i} \quad (74)$$

C Computational Appendix

To solve the model outlined in section 2 we use a policy function iteration. The algorithm proceeds as follows:

Step 0: Set an error-tolerance ϵ and form a grid M over $[0,1]$ and a grid H

over $[0,1]$. The grid M represents the financial wealth share of the agents and the grid H represents the holding of real estate of agent 1. As we have only 2 agents the equilibrium real estate holdings of agent 2 are $1-h^1$. Set an initial guess f^0 for policy and price functions.

Step 1: Given a set of policy and price functions f^{n-1} , we obtain a new set of policies and prices by the solving the system of equilibrium and first order conditions. As the short-sale constraint and collateral constraint are only occasionally binding the lagrange multipliers for the short-sale constraint and collateral constraint are differentiable at edge-cases. To circumvent this problem we use the Garcia-Zangwill trick (Zangwill and Garcia (1981)) and replace the lagrange-multipliers λ^{is} and λ^{ic} with $\lambda^{is+} = max(0, \lambda^{is})^2$, $\lambda^{is-} = max(0, -\lambda^{is})^2$, $\lambda^{ic+} = max(0, \lambda^{ic})^2$ amd $\lambda^{ic+} = max(0, -\lambda^{ic})^2$. Thus, the system of equations becomes

$$
\begin{aligned}
0 &= -\lambda_n^{ic} m \min_{s^{t+1}|s^t} q_{n-1} - \lambda_n^b r_n q_n + \left(U_n^i\right)^{\frac{\rho^i}{1-\gamma^i}-1} (1-\beta) \left(c_n^i\right)^{\frac{1-\gamma^i}{\rho^i}-1} \frac{\partial c_n^i}{\partial h_n^i} \\
&+ \left(U_n^i\right)^{\frac{\rho^i}{1-\gamma^i}-1} \beta E_{Q^i} \left[\left(U_{n-1}^i g(y^+)\right)^{1-\gamma^i} \right]^{\frac{1}{\rho^i}-1} \\
&E_{Q^i} \left[\left(U_{n-1}^i g(y^+)\right)^{-\gamma^i} g(y^+) \lambda_{n-1}^b r_{n-1} q_{n-1} \right] \\
0 &= -\lambda_t^c - \lambda_t^b q_t^b + \left(U_t^i\right)^{\frac{\rho^i}{1-\gamma^i}-1} \beta E_{Q^i} \left[\left(U_{t+1}^i g(y^+)\right)^{1-\gamma^i} \right] \\
&E_{Q_t^i} \left[\left(U_{n-1}^i g(y^+)\right)^{-\gamma^i} \lambda_{n-1}^b \frac{\partial c_{n-1}^i}{\partial \tilde{c}_{n-1}^i} \right] \\
&1-\tau \le r_t^i \le 1+\tau \\
c_n^i &= e^i + \omega^i q^h - h^i q_n - b^i p_n \\
b_n^1 + b_n^2 &= 0 \\
h_n^1 + h_n^2 &= 1 \\
c_n^1 + c_n^2 &= 1 \\
\omega_n^{i+} &= \frac{h^i q_{n-1}(\omega_n^+, s^+) + \frac{b^i}{g(s^+)}}{q_{n-1}} \\
\lambda_n^{is-} &= h_n^i \\
\lambda_n^{ic-} &= \left(b_n^i + h_n^i m \min_{s^+} q_{n-1}(\omega^+)\right)
\end{aligned}
$$

To solve for todays prices we only need the Value function U^i, consumption c^i, the financial wealth share ω^i, the holdings of housing h^i are needed and the shadow price of transaction costs.

Step 2: Prices and policy functions are updated until $||f^n - f^{n-1}|| < \epsilon$.

To solve the system of equilibrium and first order conditions one can use a solver for nonlinear equations that also handles inequality constraints. Alternatively one can use any solver for nonlinear equations but use a transformation for r_t^i that maps the whole $\mathbb{R}$ into the required bounds.

In our model