Portfolio Adjustment Costs and Asset Price Volatility with Heterogeneous Beliefs

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Abstract

This papers studies how transaction costs on stocks and bonds affects volatility and trading volume in an economy with heterogeneous beliefs. To model heterogeneous beliefs we follow Kurz (1994) and Kurz (1997) and restrict the class of beliefs to the subset of rational beliefs. Furthermore, this parameterization allows us to distinguish two cases: Market dominated by bullish beliefs and markets dominated by bearish beliefs. Although transaction costs lower trading volume in general, simulation results indicate that increasing transaction result in an increased volatility in an economy with bullish beliefs while volatility decreases in an economy with bearish beliefs.

JEL Classification Numbers: G12, G18

Keywords: Transaction Costs, Heterogeneous Beliefs, Volatility

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1 Introduction

Liquidity is a fundamental concept in finance, in more liquid markets selling or buying large quantities of assets are less costly. Empirical studies show that the relationship illiquidity and volatility is inconclusive (e.g.Amihud and Mendelson (1989) for equity markets or Bao and Pan (2013) for bond markets, i.e. in less liquid markets volatility is higher.

One way to model liquidity in financial markets is to introduce financial frictions. In particular, transaction costs are popular and a number of theoretical papers study how transaction costs affect portfolio choice and asset prices (see e.g. Aiyagari and Gertler (1991), Vayanos and Vila (1999), Vayanos (1998), Lo, Mamaysky, and Wang (2004), Constantinides (1979), Magill and Constantinides (1976), Dumas and Luciano (1991), Constantinides (1986)). Transaction costs generate a wedge between buying and selling prices, i.e. the buyer has to pay more for the asset and the seller receives less, thus decreasing the incentive to trade on markets with transaction costs. In general, these studies find that transaction costs have only a limited effect on asset prices, while portfolio choice is strongly impacted, because agents substitute assets to smooth consumption as long as other asset markets are free of financial frictions.

However, these paper focus only on transaction costs and liquidity in equity markets and ignore how liquidity on bond markets affects equity markets and vice versa. Empirically, it has been show that, while correlation between stock and bond returns are low, there exist strong volatility linkages as well as covariation of liquidity across equity and bond markets (e.g. Chordia, Sarkar, and Subrahmanyam (2005)). While Brunnermeier and Pedersen (2009) study

a model in which investors are margin constrained and they argue that there exists a strong linkage between funding liquidity, i.e. the ease with which investors can get funds, and market liquidity. Additionally, bond markets are also subject to considerable transaction costs, for example in the form of interest spreads.

In this paper we study an infinite-horizon exchange economy with heterogeneous beliefs and transaction costs on stock and bond markets to study how liquidity in each of those markets affects the other market. There exists a long literature on asset pricing with heterogeneous belief. The literature on speculative trading has been initiated by Harrison and Kreps (1978) who showed that price volatility is higher in an economy with heterogeneous beliefs. In a similar set-up, Harris and Raviv (1993) showed that heterogeneity in beliefs generate a higher trading volume. There is now a substantial literature studying the implications of heterogeneous beliefs for financial markets (see e.g. Scheinkman and Xiong (2003) for a survey).

To model heterogeneous beliefs we use the rational beliefs principle in the sense of (). The rational beliefs principle has a weaker rationality requirement than rational expectations. Under rational expectations, agents have to know the true probabilities in the economy. In an economy with rational beliefs agents' beliefs only know the empirical distribution. Furthermore, if agents at least believe that the economy is not stationary, then they will not be able to learn the true underlying probabilities. One implication of the rational beliefs principle is that agents' beliefs can only be fixed if they believe that the empirical distribution is the true distribution, i.e. they cannot be permanently optimists or pessimists.

For example, agents may expect that next-periods excess returns are higher than the historical mean. To be consistent with the rational beliefs principle agents have to change their expectations and expect excess returns that are lower than their historical mean. Furthermore, if expected positive returns are more frequent than negative returns, to be consistent with the rational beliefs principle expected excess negative returns have to be larger. We call this situation a (bullish) economy and the reverse situation, i.e. frequent lower excess returns, is called a (bearish) economy.

The rest of this paper is structured as follows. In section 2 we lay out the model, section 3 discusses the structure of beliefs. The results of the simulation exercise are discussed in section 4 and secion 5 concludes.

2   The Model

There are two representative households in the economy. Given his belief Q_t^j, a households selects portfolios and consumption plans to solve the problem

$$\max_{c_t^j, \theta_t^j, \theta_{0,t}^j} E_{Q_t^j} \left[\sum_{s=t}^{\infty} \beta^{s-t} \frac{1}{1-\gamma} \left(c_s^j \right)^{1-\gamma} \Big| H_t \right] \tag{1}$$

With β as the subjective discount factor of the household, c_t^j as the consumption of household $j \in \{1, 2\}$ at time t and H_t as the history of prices and dividends at time t. The holdings of agent j in the risky asset is denoted by θ_t^j and the one in the risk-free bond is denoted by $\theta_{0,t}^j$. The risky-asset has a net-supply of 1, while the bond is in zero net-supply.

Furthermore, as in Burnside (1998) we assume that the empirical process of

the log-dividends follows an AR(1)-process of the form:

$$x_{t+1} = (1 - \lambda)x^* + \lambda_x x_t + \rho_{t+1}^x, \quad \rho_{t+1}^x \sim \mathcal{N}(0, \sigma_x) \tag{2}$$

and $d_t = e^{x_t}$. It should be kept in mind that equation (2) is what the agents in the economy observe empirically. Under the rational beliefs principle the observed process does not have to be the true process.

In this model trading on financial markets is subject to *transaction costs*. We assume that the transaction costs are quadratic. In particular, we have

$$TC_t^S = \frac{\tau^S}{2} \left(q_t \theta_t^j - q_t \theta_{t-1}^j \right)^2, \tag{3}$$

$$TC_t^B = \frac{\tau^B}{2} \left(q_t^b \theta_{0,t}^j \right)^2, \tag{4}$$

with τ^S and τ^B representing the costs in % and q_t the price of the risky asset and q_t^b the price of the bond. Although the quadratic transaction cost is due to technical convenience, empirical studies show that transaction costs are convex or at least quadratic (see e.g. Breen, Hodrick, and Korajczyk (2002), Engle and Ferstenberg (2006) or Lillo, Farmer, and Mantegna (2003))

Additionally we assume that the transaction costs are lost, i.e. households are not reimbursed. Given the transaction costs, the budget constraint of the household is now given by

$$\widetilde{c}_t^j + \theta_t^j \widetilde{q}_t + \widetilde{\theta}_{0,t}^j q_t^b + TC_t^S + TC_t^B = \widetilde{e}_t^j + \theta_{t-1}^j (d_t + \widetilde{q}_t) + \widetilde{\theta}_{0,t-1}^j. \tag{5}$$

with e_t^j as the households' income in period t. We normalize the budget constraint by dividing it with the time-t dividend d_t. The normalized budget

equation is

$$c_t^j + \theta_t^j q_t + \theta_{0,t}^j q_t^b + TC_t^S + TC_t^B = \tilde{e}_t^j + \theta_{t-1}^j(d_t + \tilde{q}_t) + d_t \theta_{0,t-1}^j. \qquad (6)$$

The euler equations for stock and bondholdings are therefore:

$$\left(c_t^j\right)^{-\gamma}\left(q_t + \tau^S(q_t)^2(\theta_t^j - \theta_{t-1}^j)\right) = \beta E_{Q_t^j}\left[\left(c_{t+1}^j\right)^{-\gamma} d_{t+1}^{1-\gamma}\left(q_{t+1} + 1 + \tau^S(q_{t+1})^2(\theta_{t+1}^j - \theta_t^j)\right)|H_t\right]$$

$$\left(c_t^j\right)^{-\gamma}\left(q_t^b + \tau^B(q_t^b)^2(\theta_{0,t}^j)\right) = \beta E_{Q_t^j}\left[\left(c_{t+1}^j d_{t+1}\right)^{-\gamma}|H_t\right].$$

The bond and the stock-market are cleared for all $t = 1, 2, 3...$ with the following equilibrium conditions:

$$\theta_t^1 + \theta_t^2 = 1, \qquad (7)$$

$$\theta_{0,t}^1 + \theta_{0,t}^2 = 0. \qquad (8)$$

3 Beliefs

We are now turning to the discussion of the structure of beliefs. Instead of modelling the learning process we follow Kurz, Jin, and Motolese (2005a) and Kurz, Jin, and Motolese (2005b) and assume that the beliefs are part of the economic primitives. This assumption seems quite restrictive compared to models with learning. However, it can be shown that the structure presented here follows from bayesian learning (Nielsen (2007)). Furthermore, in models with learning the beliefs are still part of the state-space and the learning rule describes a transition rule and it has been argued by Jouini and Napp (2007) that in models with learning the beliefs are not 'more endogeneous' because

of the separability of learning and the optimization problem of the agent.

3.1 Individual Beliefs and the Market State of Belief

To study the rational beliefs we will introduce three different types of beliefs:

1. g_t^j denotes the state of belief of agent j as known only by the agent and we have $g_t = (g_t^1, g_t^2)$.

2. $z_t = (z_t^1, z_t^2)$ denotes the market state of belief. In our model the market state of belief and is the distribution of individual beliefs. We require that $g_t = z_t$. This model consistency condition is not recognized by an agent in the economy, however the distribution of beliefs are observable.[1]

3. $z_{t+1}^j = (z_{t+1}^{j1}, z_{t+1}^{j2})$ is the forecast of agent j of the market state of belief at the future date $t + 1$.

Let the usual state space of agent j consist of endogeneous and exogeneous variables be s^j. Here, the endogeneous state variables are the portfolios of the agents and the exogeneous variable is dividend growth. We extend this state-space by adding an additional state-variable, called the the *agent j state of belief* generated by agent j. It represents his date t subjective view of date $t + 1$ and is denoted by $g_t^j \in G^j$. With this variable we can express the conditional probably as $P(s_{t+1}^j, g_{t+1}^j | s_t^j, g_t^j)$. Furthermore, g_t^j is privately perceived by agent

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[1]This assumption might seem odd in an economy with only two agents. However, this can be remedied by the assumption that we still have a large amount of traders but only two types of traders. In such a situation households would still be able to observe beliefs of others (e.g. from available forecasts) but could associate an individual trader with her expectations.

j. We assume that g_t^j follows a process of the form

$$g_{t+1}^j = \lambda_z g_t^j + \lambda_x^{zj}(x_t - x^*) + \widetilde{\rho}_{t+1}^{g^j}, \quad \widetilde{\rho}_{t+1}^{g^j} \sim N(0, \widetilde{\sigma}_{g^j}^2) \tag{9}$$

The variable g_t^j can be interpreted as an assessment variable as in Kurz and Schneider (1996) and it can be shown that the assessment variable fully pins down the conditional beliefs of the agents. For example, an agent could believe that the empirical distribution is the true distribution. In this case the variable g_t^j has to be constant. This implies not only $\widetilde{\rho}^{g^j}$ has zero variance but also that $\lambda_x^{g^j}$, λ_{g^j} and g_t^j are all zero. The random variable $\widetilde{\rho}_t^{g^j}$ can be correlated across agents, which reflects some communication among agents (see e.g. Nakata (2007)).

In equilibrium, asset prices depend on the distribution of beliefs. Similar to the assumption that a competitive firm cannot affect prices, we assume that agents cannot affect endogenous variables, i.e. they take prices and their beliefs as given.

Next, we discuss how agents in the model forecast prices. First, we define the 'market state of belief' as a vector $z_t = (z_t^1, z_t^2)$. The model consistency condition $z_t = g_t$ is not recognized by the agents. With the prices in the economy depending now on the market state of belief we add additional uncertainty to the economy. If the usual state-variables are denoted by s_t, we can define a price-map as follows:

$$\begin{bmatrix} q_t \\ q_t^b \end{bmatrix} = \Phi(s_t, z_t^1, z_t^2). \tag{10}$$

However, with prices depending on the market-state of belief we get another implication of the extended market state of belief. To forecast the prices in the economy, households also have to forecast the market state of belief in the economy. And although all households use (10) to forecast prices, forecasts will be different because the forecast of $(s_{t+1}, z_{t+1}^1, z_{t+1}^2)$ will depend on his own state of belief g_t^j.

Because of the observability of z_t we augment the empirical distribution of the dividends with the market states of belief. We assume that the empirical distribution is an AR-process of the form

$$x_{t+1} = (1 - \lambda_x)x^* + \lambda_x x_t + \rho_{t+1}^x \tag{11}$$

$$z_{t+1}^1 = \lambda_{z^1} z_t^1 + \lambda_x^{z^1}(x_t - x^*) + \rho_{t+1}^{z^1} \tag{12}$$

$$z_{t+1}^2 = \lambda_{z^2} z_t^2 + \lambda_x^{z^2}(x_t - x^*) + \rho_{t+1}^{z^2} \tag{13}$$

$$\begin{pmatrix} \rho_{t+1}^x \\ \rho_{t+1}^{z^1} \\ \rho_{t+1}^{z^2} \end{pmatrix} \sim N \left(\begin{matrix} 0, \\ 0, \\ 0, \end{matrix} \left[\begin{matrix} \sigma_x^2, & 0, & 0 \\ 0, & 1, & \sigma_{z^1 z^2} \\ 0, & \sigma_{z^1 z^2}, & 1 \end{matrix} \right] = \Sigma \right), \text{ i.i.d} \tag{14}$$

We rewrite (11), (12) and (13) in a more compact notation, i.e. let $w_t = (x_t - x^*, z_t^1, z_t^2)$, $\rho_t = (\rho_t^x, \rho_t^{z^1}, \rho_t^{z^2})$ and denote by A the 3×3 matrix of (11), (12) and (13). Thus, we have

$$w_{t+1} = A w_t + \rho_{t+1} \, , \rho_{t+1} \sim N(0, \Sigma). \tag{15}$$

The unconditional covariance of w is denoted by $V = E(ww')$ and it is

computed as the solution of the equation

$$V = AVA' + \Sigma. \tag{16}$$

3.2 General Structure of Beliefs

We first define the *perception model* of an agent as the transition functions representing the households' conditional probability belief. Thus, to determine the households forecasts, the perception model is used. The households' beliefs can deviate from the empirical distribution, which can be interpreted as overconfidence of the households.

We denote the date $t+1$ variables as perceived by agent j as $w_{t+1}^j = (x_{t+1}^j, z_{t+1}^{1j}, z_{t+1}^{2j})$ and Γ is the stationary measure implied by (11)-(13). We also have $\Psi_{t+1}(g_{t+1}^j)$, a three-dimensional vector of date $t+1$ random variables conditional upon g_t^h.

Definition 1. *A perception model in the economy under study has the general form*

$$w_{t+1}^j = Aw_t + \Psi_{t+1}(g_t^j), \tag{17}$$

together with (11)-(13). Since $E_\Gamma[w_{t+1}|H_t] = Aw_t$, we can write (17) as follows

$$w_{t+1}^j - E_\Gamma[w_{t+1}|H_t] = \Psi_{t+1}(g_t^j). \tag{18}$$

There is documented evidence in the psychological literature that people exhibit overconfidence (see e.g. Larwood and Whittaker (1977), Svenson (1981), or Alicke (1985)). In our model households can be overconfident in the sense

that their subjective beliefs deviate from the empirical probabilities. This stands in contrast to overconfidence models such as Daniel, Hirshleifer, and Subrahmanyam (2001) which assume agents believe that a public signal contains more information than it actually does. $\Psi(g_t^j)$ is modelled by using a random variable $\eta_{t+1}^j(g_t^j)$:

$$
\Psi_{t+1}(g_t^j) = \begin{pmatrix} \lambda_g^x \eta_{t+1}^j(g_t^j) + \widetilde{\rho}_{t+1}^{xj} \\ \lambda_g^{z1} \eta_{t+1}^j(g_t^j) + \widetilde{\rho}_{t+1}^{zj1} \\ \lambda_g^{z2} \eta_{t+1}^j(g_t^j) + \widetilde{\rho}_{t+1}^{zj12} \end{pmatrix}, \widetilde{\rho}_{t+1}^j \sim N(0, \Omega_{\rho\rho}^j).
\tag{19}
$$

We can now express the perception model of agent j as follows:

$$
x_{t+1}^j = (1 - \lambda_x)x^* + \lambda_x x_t + \lambda_g^x \eta_{t+1}^j(g_t^j) + \widetilde{\rho}_{t+1}^{xj},
\tag{20}
$$

$$
z_{t+1}^{j1} = \lambda_z z_t^1 + \lambda_x^z(x_t - x^*) + \lambda_g^{z1} \eta_{t+1}^j(g_t^j) + \widetilde{\rho}_{t+1}^{zj1},
\tag{21}
$$

$$
z_{t+1}^{j2} = \lambda_z z_t^2 + \lambda_x^z(x_t - x^*) + \lambda_g^{z2} \eta_{t+1}^j(g_t^j) + \widetilde{\rho}_{t+1}^{zj12},
\tag{22}
$$

$$
g_{t+1}^j = \lambda_z g_t^j + \lambda_x^z(x_t - x^*) + \widetilde{\rho}_{t+1}^{gj}.
\tag{23}
$$

The vector $\widetilde{\rho}_{t+1}^j = (\widetilde{\rho}_{t+1}^{xj}, \widetilde{\rho}_{t+1}^{zj1}, \widetilde{\rho}_{t+1}^{zj12}, \widetilde{\rho}_{t+1}^j)$ is i.i.d. Normal with mean zero. Let $\Omega_{wg^j} = (cov(\widetilde{\rho}_{t+1}^{xj}, \widetilde{\rho}_{t+1}^{gj}), cov(\widetilde{\rho}_{t+1}^{zj1}, \widetilde{\rho}_{t+1}^{gj}), cov(\widetilde{\rho}_{t+1}^{zj12}, \widetilde{\rho}_{t+1}^{gj}))$, then the covariance matrix Ω^j is given by

$$
\Omega^j = \begin{pmatrix} \Omega_{\rho\rho}^j, & \Omega_{wg^j}^\top \\ \Omega_{wg^j}, & \sigma_{g^j}^2 \end{pmatrix}.
\tag{24}
$$

Asymmetry and fat tails in $\Psi(g_t^j)$ is introduced via $\eta_{t+1}^j(g_t^j)$. We define

$\eta_{t+1}^j(g_t^j)$ by its density $p(\eta_{t+1}^j|g_t^j)$, conditional on g_t^j, as follows:

$$p(\eta_{t+1}^j|g_t^j) = \begin{cases} \phi_1(g_t^j)f(\eta_{t+1}^j) & \text{if } \eta_{t+1}^j \geq a \\ \phi_2(g_t^j)f(\eta_{t+1}^j) & \text{if } \eta_{t+1}^j < a \end{cases}, \tag{25}$$

where η_{t+1}^j and $\tilde{\rho}_{t+1}^{g^j}$ are independent and $f(\eta) = [1/\sqrt{2\pi}]e^{-\frac{\eta^2}{2}}$ and $p(\eta_{t+1}^j|g^{t+1})$ as the conditional probability of η_{t+1}^j dependend on g^t. The functions (ϕ_1,ϕ_2) are then defined as follows:

$$\phi(g^j) = \frac{1}{1 + e^{b(g^j-a)}}, \text{ and define } G \equiv E_g\phi(g^j), \tag{26}$$

$$\phi_1(g^j) = \frac{\phi(g^j)}{G}, \phi_2(g^j) = 2 - \phi_1(g^j). \tag{27}$$

The parameter b measures the intensity of fat tails. Fat tails in the empirical distribution of returns have been attributed to resolve the equity premium puzzle (Rietz (1988), Barro (2006)) and have also been documented empirically (see e.g. Fama (1963) or Fama (1965)). While in models with rare events fat tails of the return distribution are exogeneously given they arise endogeneously in our model because of the beliefs of the agents. Furthermore, the parameter a denotes the asymmetry of the distribution of beliefs, i.e. if $a = 0$ the beliefs are symmetric and if $a \neq 0$ the beliefs are asymmetric.

Given our description above, we can define *bull* and *bear* states as follows

Definition 2. *Let Q^j be the probability belief of agent j. Then g_t^j is said to be*

*a **bear state** for agent j if $E_{Q^j}[x_{t+1}^j|g_t^j, H_t] < E_\Gamma(x_{t+1}|H_t)$;*

*a **bull state** for agent j if $E_{Q^j}[x_{t+1}^j|g_t^j, H_t] > E_\Gamma(x_{t+1}|H_t)$.*

3.3 Restriction of Beliefs

First, we define the implications of *Rational Beliefs* in the context of our simulation model:

Definition 3. *A perception model as defined in* (18) *is a Rational Belief if the agent's model* $w_{t+1}^j = Aw_t + \Psi_{t+1}(g_t^j)$ *has the same empirical distribution as* $w_{t+1} = Aw_t + \rho_{t+1}$, *i.e. the unconditional distribution of the perception model is the same as the empirical distribution.*

The interpretation of this definition is straightforward, i.e. although the conditional expectations of an agent might be different from the empirical distribution, in the long run his beliefs cannot be arbitrary. Thus, the rational beliefs principle implies that the parameters determining the beliefs have to be restricted. The following Theorem due to Kurz, Jin, and Motolese (2005a) and Kurz, Jin, and Motolese (2005b) gives us these restrictions on the beliefs:

Theorem 1. *Let the beliefs of an agent be a Rational Belief. Then the belief is restricted as follows:*

(i) *For any vector of parameters* $(\lambda_g^x, \lambda_g^z, b)$ *the Variance-Covariance matrix* Ω^j *is fully defined and not subject to choice.*

(ii) *The condition that* Ω^j *is a positive definite matrix establishes a feasibility region for the vector* $(\lambda_g^x, \lambda_g^z, b)$. *In particular, it requires* $|\lambda_g^x| \leq \sigma_x, |\lambda_g^z| \leq 1$.

(iii) $\Psi_{t+1}(g_t^j)$ *cannot exhibit serial correlation and this restriction pins down the vector*

$$\Omega_{wg^j} = [cov(\widetilde{\rho}_{t+1}^x, \widetilde{\rho}_{t+1}^{g^j}), cov(\widetilde{\rho}_{t+1}^{z1}, \widetilde{\rho}_{t+1}^{g^j}), cov(\widetilde{\rho}_{t+1}^{z2}, \widetilde{\rho}_{t+1}^{g^j})]$$

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4 Numerical Results

4.1 Calibration

To set the parameters for the simulation we follow Kurz, Jin, and Motolese (2005a) and set the parameters of the perception model as close as possible to maximum value as implied by the rationality conditions. For the dividend process, we set $x^* = 0.01773$ and $\lambda_x = -0.117$ and $\sigma_x = 0.03256$. This calibration of the empirical dividend process is consistent with the one used by Mehra and Prescott (1985). For the preferences of the households we set $\beta = 0.96$ and $\gamma = 2$. Furthermore, households' income e_t^j will be constant to 3. Setting the income constant means that households trade only because of the differences in beliefs but not because of risk-sharing.

In the non-stochastic steady-state the portfolio holdings of the agents are indeterminate, because stocks and bond are perfect substitutes at the non-stochastic steady state. Thus for the steady-state holdings of the risky asset both agents hold half the tree and no agent has debts, i.e. $\theta^j = 0.5$ and $\theta_0^j = 0$. Here we have dropped the time-subscripts to indicate that these are the values in the steady state. To ensure that the transversality conditions hold we use a penalty function. The penalty functions are of the form $\frac{\tau^{pen,b}}{2}(\theta_{0,t}^j)^2$ and $\frac{\tau^{pen}}{2}(\theta_t^j - 0.5)^2$ and we set $\tau^{pen,b} = \tau^{pen} = 0.005$.

We also consider two different economies for the simulation studies, referred to as *Economy I* and *Economy II*. For the first economy, we set $\lambda_g^x = -0.027$ and $\lambda_g^z = 0.200$ and in Economy II we set $\lambda_g^x = 0.027$ and $\lambda_g^z = -0.200$. Furthermore, the parameter b is set to $b = -8$, the parameter a to $a = -0.4$

Economic Fundamentals				
x^*	σ_x	λ_x	β	γ
0.01773	0.03256	-0.117	0.96	2

	Beliefs						
	λ_g^x	λ_g^z	λ_x^z	b	a	$\sigma_{z^1z^2}$	λ_z
Economy I	-0.027	0.200	0.900	-6	-0.25	0.9	0.7
Economy II	0.027	-0.200	0.900	-6	-0.25	0.9	0.7

Table 1: The parameters of the economic fundamentals and the beliefs of the agents.

and we set $\lambda_x^z = 0.9$. Finally, the correlation of beliefs $\sigma_{z^1z^2}$ is set to 0.9. All parameters are summarized in table 1.

Because of Definition 2 the difference between the two economies is the frequency of bull and bear states. Bear states are more frequent in Economy I, whereas in Economy II bull states are more frequent.

For the simulation study the transaction costs on bonds and stocks are in the range of 0% and 1% and costs are increased by 5 basis points.

An important aspect of our study is the trading volume in the economy and we will use the trading volume as a proxy for liquidity on the market, i.e. a lower trading volume implies a less liquid market.[2] The trading volume in

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[2]Although the trading volume is only a crude measure for liquidity other liquidity measures based on the market microstructure literature imply a positive relation ship between the trading volume and the liquidity of financial markets. Furthermore, other empirical measures of liquidity rely incorporate trading volume.

the stock and bond market is defined as follows[3]:

$$V = \frac{1}{T} \sum_{t=1}^{T} |\theta_t^j - \theta_{t-1}^j|, \tag{28}$$

$$V^b = \frac{1}{T} \sum_{t=1}^{T} |\theta_{0,t}^j|. \tag{29}$$

The economy is approximated using a second order perturbation using the software pertsolve by Jin (2003), details of the computational model are given in Appendix A.

To calculate trading volume and prices we do a Monte Carlo Simulation. In particular we simulate one path with 100.000 periods

4.2   Simulation Results

4.2.1   Transaction Costs on Stocks

First, we are studying the impact of transacion costs on the stock market. The main results are as follows:

Result 1. *Higher transaction costs reduce trading volume in stocks and bonds, whereas for the volatility of the Price/Dividend-ratio and interest rates we have the following:*

- *The volatility of the Price/Dividend-ratio increases in Economy I and is nearly unaffected in Economy II;*

- *The volatility of the interest rates decreases in Economy I and increases in Economy II.*

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[3]see Lo and Wang (2010) for various definitions of trading volumes
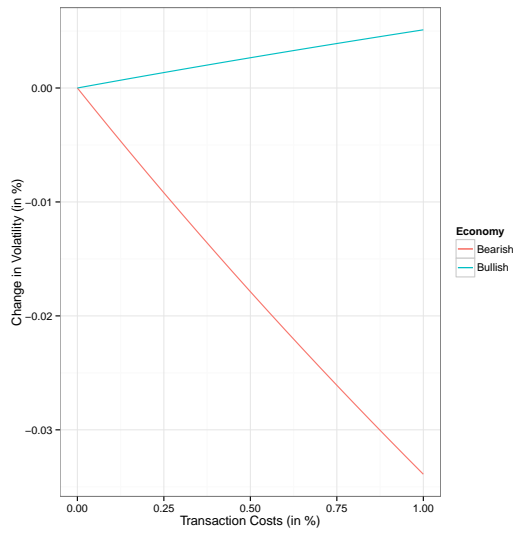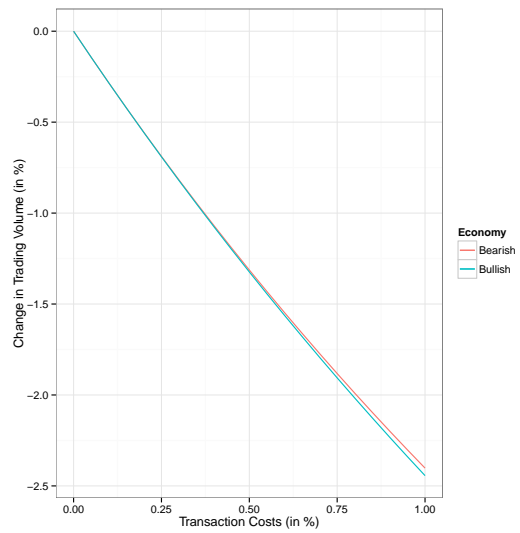
(a) P/D-Ratio
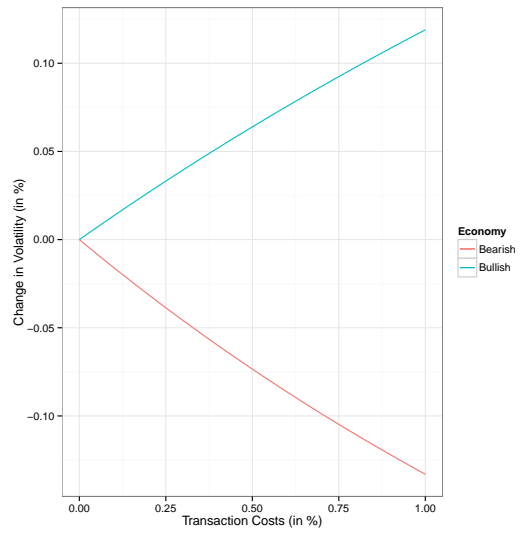
(b) Interest Rate

Figure 1: Transaction Costs on Stocks
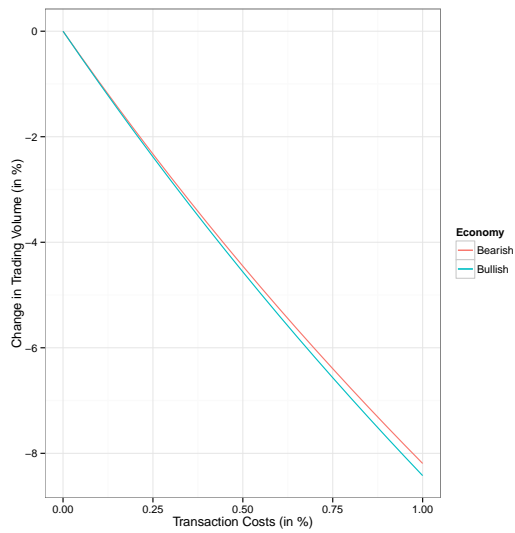
(a) Trading Volume Assets

(b) Trading Volume Bonds

Figure 2: Transaction Costs on Stocks

Figure (1) shows the impact of transaction costs on stock market on the volatility of the Price-Dividend Ratio and the Interest Rate. One can clearly see that the impact of transaction costs differs between between a bullish and a bearish economy. In particular, in a bearish economy volatility increases while in a bearish economy volatility decreases. Yet the changes in volatility are only very small, i.e. less than 0.2% for Price/Dividend-Ratio and interest rates. On the other hand, as can be seen from figure (2), the trading volume of both stocks and bonds decline with increasing transaction costs. Thus liquidity drops on both markets and in both economies. Furthermore, the change is substantially larger, i.e.trade in risky asset drops by 8% with a transaction cost of 1%, while trading volume on the bond market drops by about 2.5% with a 1% transaction cost.

The decrease in trading volume and the simultaneous increase in volatility in Economy I is qualitative similar to empirical results. Empirical estimations of the elasticity of trading volume with transaction costs on stock markets varies from 0 (Hu (1998)) to −1.7 (Lindgren and Westlund (1990)). With an elasticity of −1 meaning that if transaction costs increase by 50% the trading volume has to decrease by 50% as well. Thus, our model produces a reduction in trading volume which is consistent with the empirical evidence. Furthermore, the observed change in the volatility is within the empirically observed range. While Hu (1998) finds that transaction costs do not affect volatility Hau (2006) observes that increasing the tick size on the french stock market increases transaction costs by 20% and volatility by 30%.

As the households' income are fixed every period, they face no idiosyncratic risk. However, differences in beliefs across states can be interpreted

as 'prefefence shocks' (see e.g. Cogley, Sargent, and Tsyrennikov (2014)) and households trade on financial markets to hedge against preference shocks induced by heterogeneity of beliefs.

We can define the net-returns, i.e. the returns after transaction costs, a household gets from holding the risky asset as follows:

$$r_t^{net} = \frac{q_{t+1} + v_{t+1} - \tau^s \left(\theta_{t+1}^i - \theta_{t+1}^i \right) q_{t+1}^2}{q_t + \tau^s (\theta_{t+1}^i - \theta_t^i) q_t^2}, \tag{30}$$

It is easy to see that with rising transaction costs, the net-returns from holding the risky asset declines and thus households may be tempted to trade in the bond market to hedge the preference shocks. Yet, as trading volume declines in both markets, households forego the opportunity to use the bond market for risk-sharing. This is in contrast to the case with idiosyncratic *income* risk and homogeneous beliefs (see e.g. Heaton and Lucas (1996)). In an economy with idiosyncratic income risk and homogeneous beliefs, households shift their trading onto the bond market to hedge against labor income risk.

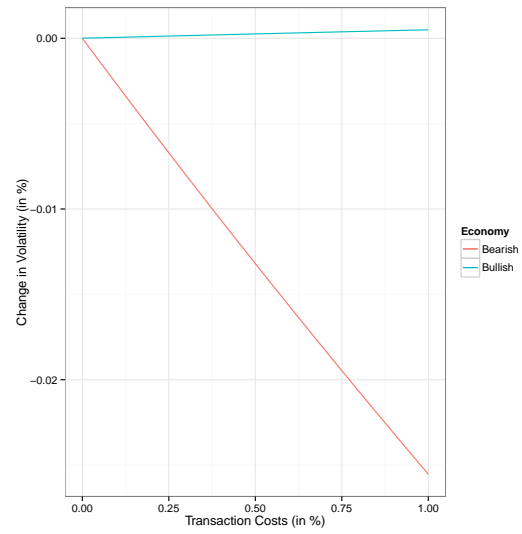### 4.2.2 Transaction Costs on Bonds

We are now turning to the discussion on the effects of transaction costs on the bond market and we have the following results:

Result 2. *Transaction costs on the bond market results in a decreasing trading volume, whereas for the volatility of the price/dividend-ratio and the interest rates we have:*

- *The volatility of the price/dividend-ratio increases in Economy I and decreases in Economy II;*
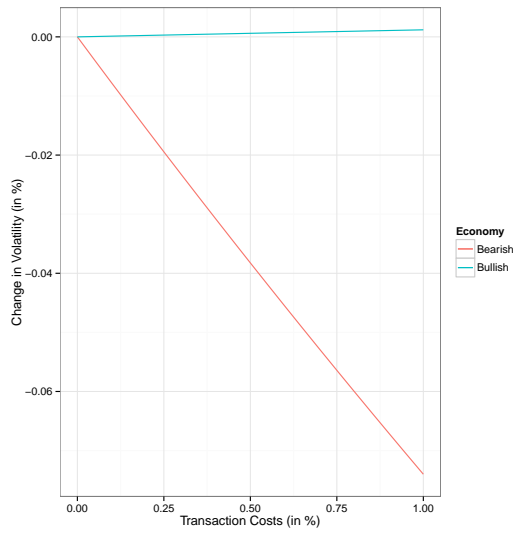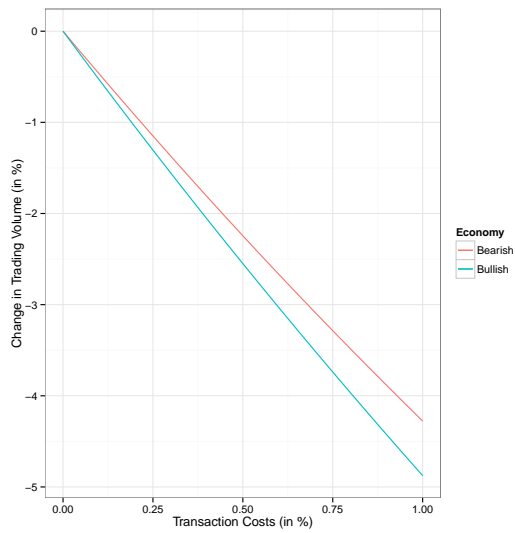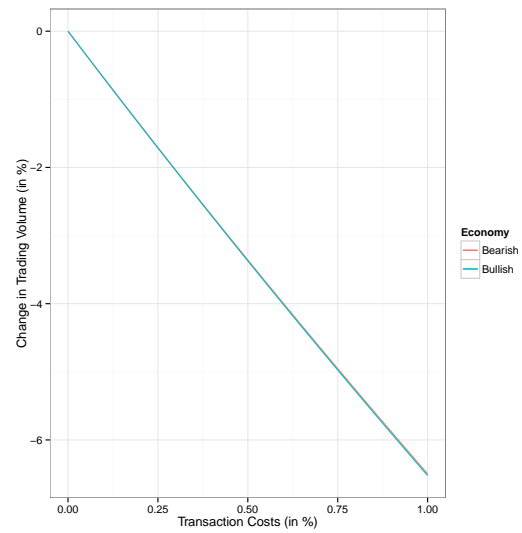
(a) P/D-Ratio

(b) Interest Rate

Figure 3: Transaction Costs on Bonds

(a) Trading Volume Assets

(b) Trading Volume Bonds

Figure 4: Transaction Costs on Bonds
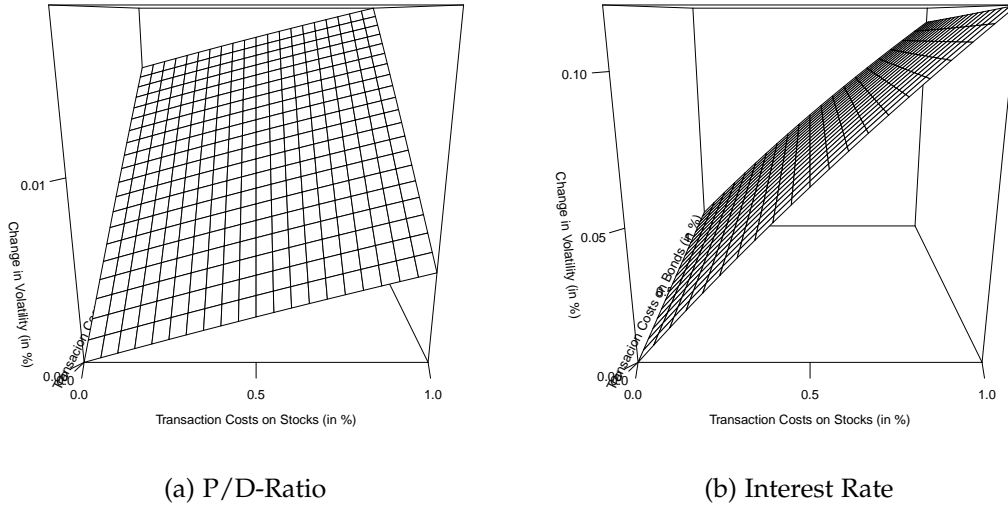
(a) P/D-Ratio (b) Interest Rate

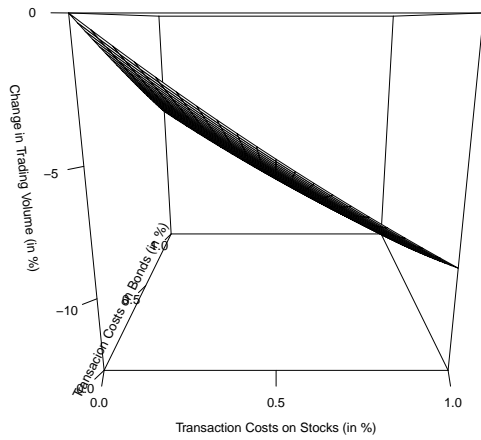Figure 5: Simultaneous transaction costs on stock and bond markets

- *The volatility of the interest-rate decreases in Economy I and increases in Economy I.*

In figure (3) we depict the volatility of the price/dividend ratio and interest rate if there are transaction costs on the bond market only. Again, volatility in the bearish economy increases but this time the increase is negligible, while volatility in the bullish economy increases. Furthermore, as shown in figure (4), trading volume and thus liquidity drops in both markets.
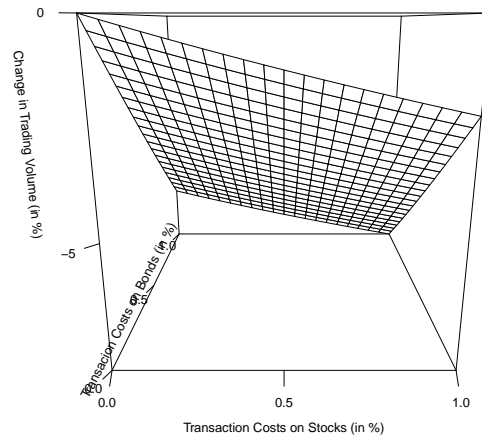
Again, liquidity reduces across both markets if one market is subject to transaction costs.

4.3 Simultaneous Transaction Costs on Stocks and Bonds

So far, the discussion was only about the cases in which there were transaction costs only on one market. Now, we are turning to the case in which there

(a) Trading Volume Stocks   (b) Trading Volume Bonds

Figure 6: Simultaneous transaction costs on stock and bond markets

are transaction costs on both markets, i.e. the stock and the bond market. As only in the bearish economy volatility increased with transaction costs, which is consistent with empirical evidence, we focus on the bearish economy in this section. The key results of this exercise are as follows:

Result 3. *Transaction costs on stock and bond market has the following effects:*

1. *Increasing transacion costs on any market always reduces the trading volume.*

2. *In Economy II transaction costs bonds always reduces the volatility of the P/D-Ratio whereas transaction costs on stocks reduces only the volatility of the P/D-Ratio if there are positive transaction costs on bonds. Furthermore, the volatility of the interest rate does not always decrease.*

As it can be seen from figures (5) and (6) transaction costs on both markets exacerbate each other. In particular, the volatility of the price/dividend ratio

is highest when both transaction costs are at 1%.

Thus, to conclude, our numerical results indicate that reducing liquidity in one market does affect the other market as well.

4.4 Comparative Statics

Thus far, we have seen that the introduction of a transaction costs have ambiguous effects on market volatility, i.e. the volatility can go up or down depending whether the liquidity effect or the speculative effect dominates. Thus we are now investigating how the volatility depends on the parameters for the beliefs. The key results of this exercise are as follows:

Result 4. *A change in the beliefs parameters (a, b) has the following effects on the volatility of the P/D-ratio:*

1. *in Economy II no parameter has a monotonic effect on the volatility of the P/D-ratio.*

We first focus on the asymmetry of beliefs, i.e. different values for a. In figures 7 and 7 show the volatility of the P/D-ratio, while figures 8 and 8. First, we see that changing the parameter a from $a = -0.40$ to $a = -0.20$ does not affect volatility at all. However changing a from $a = -0.40$ to $a = -0.60$ reduces the change in volatility. A value of $a = -0.60$ implies that beliefs are more skewed than with the value of $a = -0.40$. However, with more skewed beliefs the need to share preference shocks induced by heterogeneity in beliefs is increased. With a stronger motivation to trade, transaction costs have less of an impact on trading and thus on volatility.
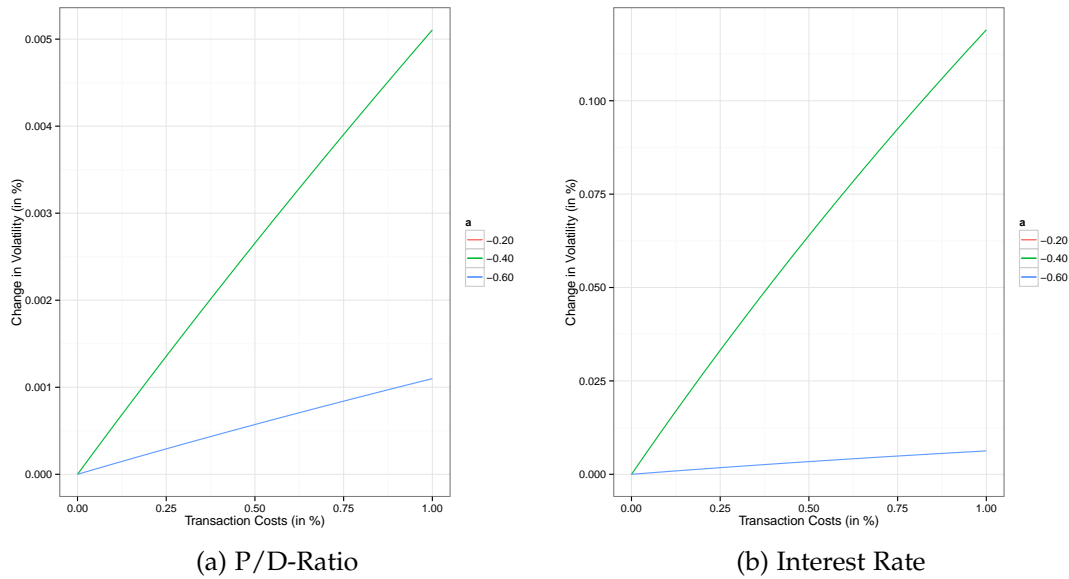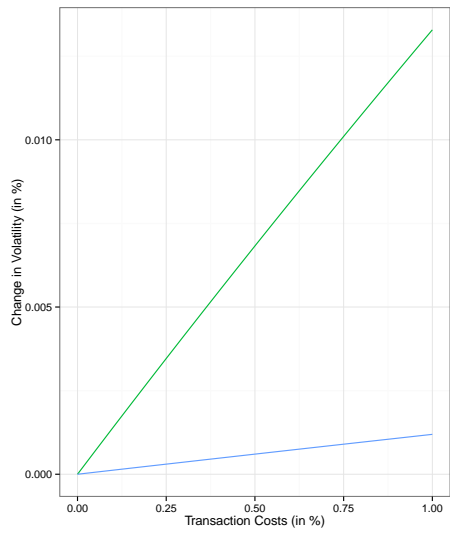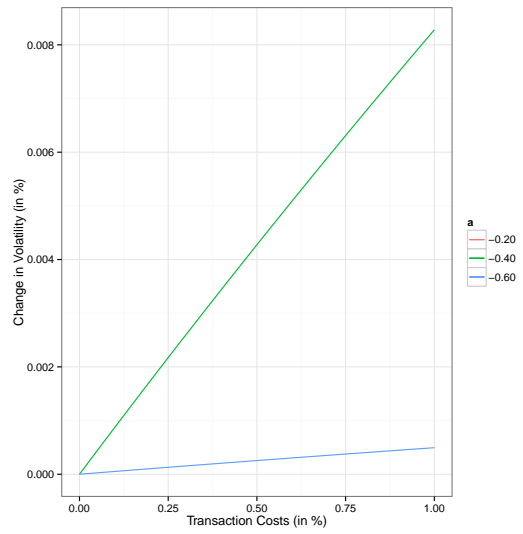
(a) P/D-Ratio

(b) Interest Rate

Figure 7: Transaction Costs on Stocks and Asymmetry of Beliefs

The volatility of the P/D-ratio under different parameters for b is shown in figures (9) and (10) In both economies a change in the intensity of the fat tails has a non-monotonic effect on the volatility. A very low b means that the households' beliefs have large fat tails thus gives the households more incentive to speculate on a favorable outcome. In an economy where bullish expectations are rare, the households could gain a lot from the dividends. Whereas in an economy in which bearish expectations are rare households will have an incentive to sell the stock to not get hit by a crash. As the net-supply of the stock is positive the equilibrium price has to adjust. In particular it has to become smaller so that the agents will hold the asset, hence the volatility increases.
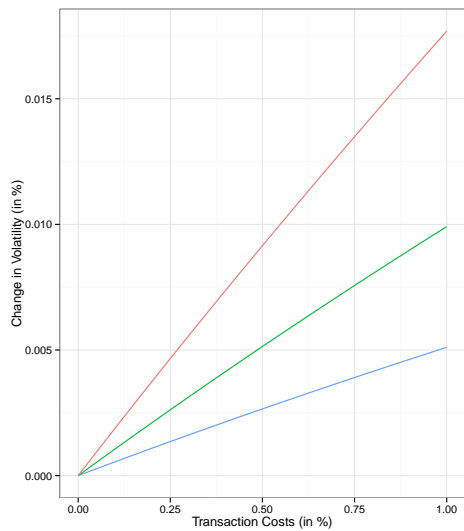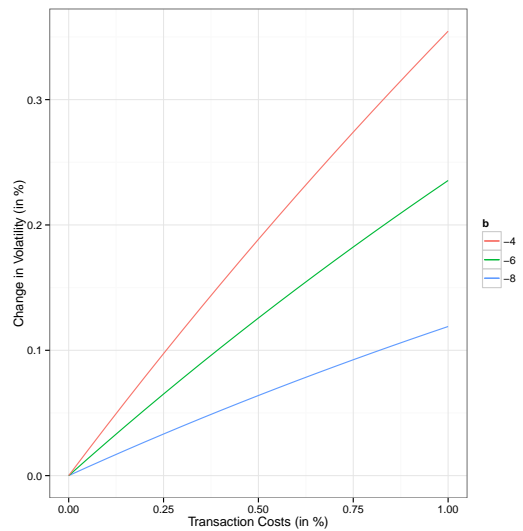
(a) P/D-Ratio

(b) Interest Rate

Figure 8: Transaction Costs on Bonds and Asymmetry of Beliefs

(a) P/D-Ratio

(b) Interest Rate

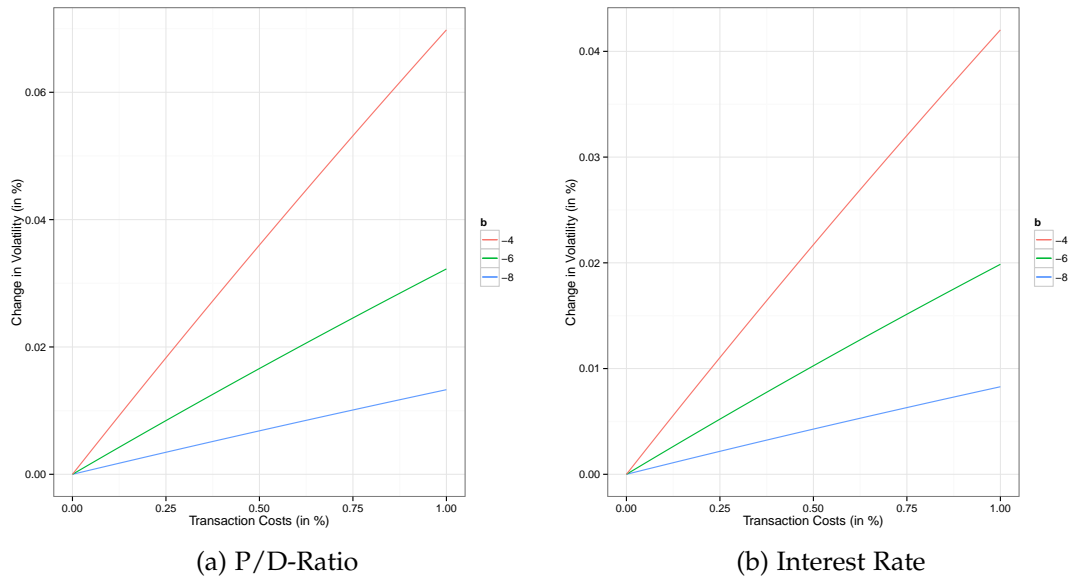Figure 9: Transaction Costs on Stocks and Intensity of Fat Tails

(a) P/D-Ratio (b) Interest Rate

Figure 10: Transaction Costs on Bonds and Intensity of Fat Tails

5 Conclusion

In this paper we studied the impact of transaction costs on stock and bond purchase on asset price volatility in an economy with heterogeneous beliefs. The model was able to replicate some important empirical regularities. In particular increasing transaction costs implied a lower trading volume but a higher asset price volatility.

Furthermore, even moderate transaction costs on stock and bond purchases resulted in a substantial drop in trading volume. This large drop implies that income from transaction taxes as proposed for example by the European Union - which is supposed to tax not only stocks but also bonds and derivatives - may have overall negative effects. In particular a larger stock market volatility may make it more difficult for firms to raise capital which affects growth negatively.

The model in this paper can be extended into several directions. First, the

model in this paper has only one stock market. In the presence of several stock markets one would be able to study the migration of traders from a financial market with a transaction costs (e.g. because a country introduces transaction taxes) towards a stock market without transaction costs. This may be an issue if smaller countries introduce a transaction tax. For example Umlauf (1993) discusses the experience of the swedish stock market where a large fraction of the trade in stocks migrated to London.

Furthermore, the model presented in this paper is a pure asset-pricing model and the connection between the financial side of the economy and the real side of the economy is not explored here, in particular the effect of a transaction costs on business cycles. Studying the implications of transaction costs on financial markets on the business cycle would also provide another fruitful direction of research.

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A Computational Appendix

We are describing now the simulation methodology to retrieve the stationary distribution. First, we are calculating a second-order approximation of the economy (see e.g. Jin and Judd (2002)) with the program pertsolv by Jin (2003). Then, we are doing a Monte Carlo simulation of the system until the averages and variances converge.

The steady-state prices for the bond and risky asset are given by

$$
\begin{aligned}
q_{ss}^b &= \beta \exp{-\gamma x^*}, \\
q_{ss} &= \frac{\beta \exp{(1-\gamma)x^*}}{1 - \beta \exp{(1-\gamma)x^*}}.
\end{aligned}
$$

The short-sale constraint $\theta \geq \bar{\theta}$ and $\theta_m \geq \bar{\theta}_m$ are approximated with a quadratic penalty function. This penalty function ensures that the No-Arbitrage condition is satisfied. The penalty functions are: $\frac{\tau_S}{2}(\theta_t^h)^2$ and $\frac{\tau_M}{2}(\theta_{m,t}^h)^2$ with $\tau_M = \tau_S = 0.001$. Thus, the modified Euler equations become

$$
\begin{aligned}
\frac{1}{p_t^B}\exp\{-\gamma^h c_t^h\} + \tau_M(\theta_{m,t}^h) &= E_{Q_t^h}[\exp\{-\gamma^h c_{t+1}^h\}\frac{\beta}{p_{t+1}^B}|I_t], \\
\frac{q_t}{p_t^B}\exp\{-\gamma^h c_t^h\} + \tau_S(\theta_t^h) &= E_{Q_t^h}[\exp\{-\gamma^h c_{t+1}^h\}\beta\frac{q_{t+1}+v_{t+1}}{p_{t+1}^B}|I_t].
\end{aligned}
$$

For the perturbation method, we simulate now the following set of equations:

$$x_{t+1} = (1 - \lambda_x)x^* + \lambda_x x_t + \lambda_g^x \eta_{t+1}^h(\widetilde{u}_{t+1}^h, \epsilon) + \epsilon \widetilde{\epsilon}_{t+1}^{v^h},$$

$$z_{t+1}^{h1} = \lambda_z z_t^1 + \lambda_x^z(x_t - x^*) + \lambda_g^z \eta_{t+1}^h(\widetilde{u}_{t+1}^h, \epsilon) + \epsilon \widetilde{\epsilon}_{t+1}^{z^{1h}},$$

$$z_{t+1}^{h2} = \lambda_z z_t^2 + \lambda_x^z(x_t - x^*) + \lambda_g^z \eta_{t+1}^h(\widetilde{u}_{t+1}^h, \epsilon) + \epsilon \widetilde{\epsilon}_{t+1}^{z^{2h}},$$

where ϵ is a scaling-factor. At the steady-state the scaling factor is $\epsilon = 0$. For the Monte Carlo Simulation we set the scaling-factor to $\epsilon = 0.0001$.

We compute the equilibria with a perturbation method. We declare a solution an equilibrium if: (i) a model is approximated by at least second order derivatives, (ii) errors in market clearing conditions and Euler equations are less than 10^{-6}.

For the Monte Carlo Simulation we proceed as follows. In each round we will calculate prices, consumption and new portfolios given the portfolios of the previous round and this periods dividend shocks.

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