# Macroeconomic Implications of the Long-Run Decline in Real Interest Rates*

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**Abstract**

Since the early 1980s the U.S. economy has been going through a long-run decline in real interest rates. At the same time, the following developments took place: house prices and mortgage volume increased strongly, the share of expenditures for education in GNP grew steadily, the U.S. experienced large capital inflows, and labor income inequality rose substantially. We use a calibrated macro model to argue that the long-run decline in real interest rates can account for a substantial part of these developments.

**Keywords**: Housing Market,
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# 1 Introduction

Since the early 1980s the U.S. economy has been going through a long-run decline in real interest rates. More specifically, in the first half of the 1980s real interest rates on treasuries, high-grade corporate bonds, mortgages, and certificates of deposit started to fall gradually and this decreasing trend has continued for at least three decades. At the same time, the U.S. economy has experienced a number of remarkable developments. First, the U.S. residential housing market was booming. Despite the 2007 housing bust, house prices are currently almost 40% higher than 1985 in real terms and rose by more than 30% relative to rents. As a result, the market value of residential land increased from 40% of GNP in 1985 to 50% in 2015. Second and related, the U.S. economy went through a great mortgage boom in the same period. The mortgage-to-GNP ratio rose from 30% in 1985 to more than 70% at peak in 2007 and is still above 50%. In fact, mortgage volumes grew faster than house prices so that the aggregate loan-to-value ratio increased from 40% to about 60% in 2015. Third, education spending in the U.S. has been rising steadily since 1985. More specifically, the share of total expenditures for education amounted to 6% of GNP in 1985 and rose by ca. 20% during the following three decades. Forth, the U.S. current account turned negative in 1983. Since then, the U.S. has experienced large capital inflows amounting to about 60% of GNP in cumulative total. Fifth, as the literature documents (e.g. Heathcote, Perri, and Violante, 2010; Heathcote, Storesletten, and Violante, 2010; Piketty and Saez, 2003; Piketty, Saez, and Zucman, 2018), labor income inequality increased substantially. In this paper, we ask two questions. First, how does an economy react to a long-run decline in real interest rates? Second, to what extent can the long-run decline in real interest rates observed in the U.S. quantitatively explain these recent developments?

To address these questions, we develop a macro model with a housing sector and conduct a quantitative analysis of the long-run decline in real interest rates on a calibrated version of the model economy. We consider an open economy with perfect international capital mobility. Consequently, the domestic real interest rate is equal to the world interest rate whose time path is exogenously given. In the model, households can buy consumption goods, save in a risk-free asset, rent or invest in housing space, and invest in human capital. Housing investment and human capital investment are subject to uninsurable idiosyncratic risk. Households can also borrow against their housing wealth and default on their collateralized mortgage debt, in which case they lose their housing investment. To put it in a nutshell, households make a consumption-saving decision, a portfolio choice decision and a default decision. Household are ex-ante heterogeneous with respect to age. We model the life as cycle as follows: Young households age stochastically, while old households die stochastically and are replaced by a young household. We close the model assuming a fixed supply of housing (land) and an aggregate production function that displays constant returns to scale with respect to physical capital and human capital. Financial intermediaries transform households' risk-free asset holdings into physical capital and offer mortgages in competitive markets.

For the quantitative analysis we consider a version of the model that is calibrated to the U.S. economy in 1985. When we feed the gradual decline in the real interest rates between 1985 and 2015 into the calibrated model economy, we find that the model simulations are qualitatively in line with most of the important developments of U.S. economy during the last four decades. More specifically, our results can be summarized as follows. First, the model predicts recent trends in the U.S. housing market very well. House prices and mortgage volume rise, but mortgage volume rises faster than house prices so that loan-to-value ratios (leverage) increase. While the long-run increase in both the price-rent ratio and the market value of housing are entirely captured, the simulations account for half of the boom in mortgage. Second, the computational experiment matches the time-series patterns of U.S. educational expenditures quite well, with a human capital investment rate that is about 20% higher in 2015 than 1985. This human capital deepening is key for the increase in GNP by 7.5% til 2015 that the model predicts. Third, in the capital market the simulations overestimate actual capital inflows and investments in the physical capital stock. However, the experiment is able to quantitatively explain the declining trend in national savings. Finally, in our calibrated model economy, the long-run decline in interest rates can also explain a substantial fraction of the observed increase in labor income inequality during the last decades.

## 1.1 Related Literature

The model used in this paper combines the tractable incomplete-market model with human capital developed in Krebs (2003) with a tractable model of housing and mortgage markets with collateralized default along the lines of Jeske, Krueger, and Mitman (2013). In our framework, individual households solve a Merton (1969)-type consumption-saving and portfolio problem, which is augmented by a default decision. The tractability of the model derives from the result that consumption choices are linear in total wealth and that portfolio and default choices are independent of wealth. This property allows us to solve for the general equilibrium without knowledge of the endogenous wealth distribution. The tractability of the framework makes it an useful vehicle for the analysis of the interaction between housing markets and human capital formation. Therefore, the model might be of independent interest.

On the substantive side, this paper contributes to a section of the growing macro-housing literature that quantitatively investigates the role of interest rates for housing and mortgage markets. In this respect our paper is closely related to the work of Kiyotaki, Michaelides, and Nikolov (2011), Sommer, Sullivan, and Verbrugge (2013), Favilukis, Ludvigson, and Van Nieuwerburgh (2017), and Garriga, Manuelli, and Peralta-Alva (2018), who all study the effect of fundamentals, including interest rates, on housing market outcomes such as house prices, rents or ownership rates as well as macroeconomic aggregates. Unlike previous studies, we do not only focus on housing and mortgage markets, but we explicitly take the implications for human capital formation, labor income inequality, and international capital flows into account.

# 2 Model

## 2.1 Households

**Demographics.** Time $t$ is discrete and open ended: $t = 0, 1, \ldots$. The economy is populated by a unit mass of households. Households belong to one of two age groups: $z = y$ (young households) and $z = o$ (old households). Young households age stochastically and the probability of moving from $z = y$ to $z = o$ is denoted by $\rho$. Old households die stochastically and the probability of death is denoted by $q$. An old household who dies is replaced by a young household. We assume that the demographic structure of the population is stationary. Thus, in every period there are $\frac{q}{q+\rho}$ young households and $\frac{\rho}{q+\rho}$ old households.

**Preferences.** Households derive utility from consumption of two goods: a standard consumption good and housing services. Households have identical preferences that allow for a time-additive expected utility representation with pure discount factor $\beta$ and one-period utility function

$$u(c_t, c_{xt}) = \ln c_t + \nu \ln c_{xt} \tag{1}$$

where $c_t$ is consumption of the standard good in period $t$ and $c_{xt}$ is consumption of the housing service in period $t$.

**Budget.** Consider a household born in period $t = 0$. This household begins life with an initial holding of financial assets, $a_0$, an initial holding of mortgage debt $m_0$, initial ownership of housing $x_0$, and an initial stock of human capital (skills), $h_0$. Households can invest in the risk-free financial assets, risky housing and risky human capital. In addition, households who invest in housing can take out (additional) mortgage debt. Thus, for a household born in period $t = 0$, the budget constraint reads:

$$
\begin{aligned}
(2) \qquad & c_t + rr_{xt}c_{xt} + a_{t+1} + m_{t+1} + P_t x_{t+1} + i_{ht} \; = \\
& (1 + (1-\tau)r_{ft})a_t + (1 + (1-\tau)r_{mt})m_t + (1-\tau)rr_{ht}h_t \; + \\
& \left[(1-\tau)rr_{xt} + (1+\epsilon_t)P_t - \tau((1+\epsilon_t)P_t - P_{t-1})\right] x_t \\
& h_{t+1} \; = \; (1 - \delta_h(z_t) + \eta_t)h_t + i_{ht} \\
& x_{t+1} \geq 0 \;,\; m_{t+1} \leq 0 \;,\; h_{t+1} \geq 0 \;,\; a_{t+1} \geq 0
\end{aligned}
$$

In (2) variables are defined as follows. The variable $a_t$ denotes financial asset holding in period t, $m_t$ stands for the mortgage holding, $x_t$ is the housing stock owned by the household, $h_t$ denotes the household stock of human capital and $i_{ht}$ the investment in human capital. Note that this budget constraint implicitly lumps together general human capital (education, health) and specific human capital (on-the-job training). Further, $r_f$ is the risk-free rate paid on financial assets, $r_m$ denotes the mortgage rate (if no default occurs), $rr_h$ is the rental rate of human capital (the wage rate per unit of human capital), $rr_x$ is the rental rate of housing (the housing rent per unit of housing), $P$ is the aggregate (average) price of one unit of housing. Note that these rental rates and prices in general depend on time $t$. The time-invariant parameter $\tau$ denotes a flat income tax rate. Finally, $\delta_h(z)$ is the age-dependent depreciation rate of human capital and $\eta$ and $\epsilon$ are idiosyncratic shocks to human capital and the individual value of housing, respectively, that make human capital investment and housing investment risky.

The shocks to human capital, $\eta$, and housing, $\epsilon$, are i.i.d. across households (idiosyncratic shocks) and i.i.d. over time (no predictability). We regard $\eta$ as uninsurable labor income risk. Positive realizations of the human capital shock $\eta$ can be interpreted as internal promotions or upward movements in the labor market, while examples for negative ones are a decline in health (disability) and losses of firm- or sector-specific human capital due to worker displacements (cf. Krebs, 2003). We interpret $\epsilon$ as regional shocks or shocks to the local housing market. Finally, we assume that the random variables $\eta$ and $\epsilon$ are independent of each other. Note that the housing price index $P_t$ is endogenous in the model, whereas the idiosyncratic shock $\epsilon_t$ is exogenous.

The budget constraint (2) assumes that households do not care whether they live in a rented home or a owner-occupied home. Thus, the decision of housing consumption, $c_x$, and the decision of housing investment, $x$, are separated from each other. Budget constraint (2) also assumes that the consumption good can be transformed one-to-one into physical and human capital. Besides, it allows households to save and dissave in both types of assets, physical as well as human capital. In particular, for tractability reasons (see below) it does not impose any restriction on the ability of households to decumulate human capital. However, in the calibrated model economy used for our quantitative analysis, equilibrium human capital investment is always non-negative: $i_{ht} \geq 0 \; \forall s_t$. Only when young households (stochastically) retire, we allow workers to transform a fraction of $1 - \delta_h(o)$ of their human capital into financial assets. By generating additional income, this shortcut intends to depict pensions in our model in a simple way.

For a household born in period $t > 0$, the budget constraint is still given by (2). However, the index $t$ has to be replaced by the pair $(t, j)$, where $j$ denotes the period in which the household was born and $t - j$ is the age of the household. For notational simplicity, we focus attention on the household problem for households born in $t = 0$.

**Decisions.** In each period $t$, an individual household makes a consumption choice $(c_t, c_{xt})$ and an investment choice $(a_{t+1}, x_{t+1}, m_{t+1}, h_{t+1})$. In addition, households make a default decision after having observed the idiosyncratic house price shock $\epsilon_t$. In case of default, all mortgage debt is canceled ($m_t = 0$) and the entire housing stock is lost due to foreclosure ($x_t = 0$). Following Jeske, Krueger, and Mitman (2013), we assume that there are no future consequences of default; that is, we assume that subsequent to the default decision the household has full access to the mortgage and housing market, and there is no garnishment of wages. However, we assume that in the period of default a cost $\gamma \; m_t$ occurs, where $\gamma$ is an exogenous parameter. We can interpret $\gamma$ narrowly as the legal cost of default. However, we can also interpret $\gamma$ more broadly as a crude way of modeling the entire cost of default including future wage garnishment and/or future exclusion from financial markets. These assumptions imply, as argued below, that households will choose to default whenever the net relief of mortgage debt is greater than the (cum-rent) value of the house: $-(1 - \gamma + r_{mt})m_t > [rr_{xt} + (1 + \epsilon_t)P_t]x_t$.

Households take the price sequence $\{r_{ft}, r_{mt}, rr_{xt}, rr_{ht}, P_t\}$ as given. For a given price sequence and given initial state $(a_0, x_0, m_0, h_0)$ and $(z_0, \eta_0, \epsilon_0)$, an individual household born in period $t = 0$ chooses a plan, $\{c_t, c_{xt}, a_t, x_t, m_t, h_t\}$, that maximizes expected lifetime utility subject to the sequential budget constraint (2), where expected lifetime utility associated with a consumption plan, $\{c_t, c_{xt}\}$, is defined by the one-period utility function (1), the discount factor, $\beta$, and the underlying probabilities and choices. Here $\{c_t, c_{xt}, a_t, x_t, m_t, h_t\}$ is a sequence of functions mapping histories $(z^t, \eta^t, \epsilon^t)$ into choices $(c_t, c_{xt}, a_{t+1}, x_{t+1}, m_{t+1}, h_{t+1})$.

For households born in periods $t > 0$, the household decision problem is defined accordingly. We denote the plan chosen by a household born in period $j$ by $\{c_{jt}, c_{x,jt}, a_{jt}, x_{jt}, m_{jt}, h_{jt}\}$.

## 2.2 Production and Housing Supply

**Production.** The non-housing consumption good is produced by one representative firm using a Cobb-Douglas production function with two input factors, physical capital and human capital:

$$(3) \qquad Y_t = BK_t^{\alpha} H_t^{1-\alpha}$$

In (3) the variable $Y_t$ denotes output produced in period $t$, $K_t$ is the aggregate stock of physical capital in period $t$, and $H_t$ is the aggregate stock of human capital in period $t$. Further, $B$ is a parameter measuring total factor productivity and $\alpha$ is a productivity parameter that is equal to capital's share in output.

The representative firm rents physical capital and human capital in competitive markets at rental rates $rr_k$ and $rr_h$, respectively. Note that $rr_h$ is simply the wage rate per unit of human capital. In each period $t$, the representative firm solves the static profit maximization problem:

$$(4) \qquad \max_{K_t, H_t} \left\{ BK_t^{\alpha} H_t^{1-\alpha} - rr_{kt} K_t - rr_{ht} H_t \right\}$$

The first-order conditions associated with the profit maximization problem (4) yield rental rate functions

$$(5) \qquad \begin{aligned} rr_k\left(\tilde{K}_t\right) &= \alpha B \tilde{K}_t^{\alpha-1} \\ rr_h\left(\tilde{K}_t\right) &= (1-\alpha) B \tilde{K}_t^{\alpha} \end{aligned}$$

where $\tilde{K} = K/H$ is the capital-to-labor ratio chosen by the representative firm.

**Housing Supply.** Housing is in fixed supply (land) and the aggregate supply of housing is normalized to one: $X_t = 1$. There is a large number of housing units that are subject to idiosyncratic shocks, $\epsilon_t$, to the stock (value) of an individual housing unit. Thus, if $P_t$ is the aggregate price of housing in the economy, then $(1+\epsilon_t)P_t$ is the price of one individual housing unit with shock realization $\epsilon_t$. The housing price index $P_t$ is endogenous, whereas the idiosyncratic component $\epsilon_t$ is exogenous.

We assume that one unit of the housing stock generates one unit of housing consumption services. Consequently, housing consumption services are also in fixed supply: $C_{xt} = 1$.

## 2.3 Financial Intermediation

**Mortgages.** Financial intermediaries borrow at the risk-free rate, $r_f$, and use the funds to make mortgage loans to households. Mortgage lending generates a cost of $\Delta$ per unit of mortgage. In the case of default, the mortgage claim is written off and the financial intermediary receives a fraction $\kappa < 1$ of the mortgage value from the liquidation of the housing stock associated with the mortgage (foreclosure). The parameters $\Delta$ and $\kappa$ are exogenous.

Financial intermediaries can fully diversify mortgage risk and mortgage markets are perfectly competitive. Thus, financial intermediaries make zero (expected) profit, which implies,

$$(1 - \pi_{dt}(1-\kappa))\,(1+r_{mt}) \;=\; 1 + r_{ft} + \Delta \tag{6}$$

where $\pi_{dt}$ is the probability of mortgage default. This default probability is endogenously determined by the default decision of households.

**Physical Capital.** Financial intermediaries can also transform one unit of financial capital into $\phi_t$ units of physical capital. Zero profit requires

$$(1+r_{ft}) = \frac{\phi_{t-1}}{\phi_t}(1-\delta_k) + \phi_{t-1} rr_{kt} \tag{7}$$

where $\delta_k$ is the rate of physical depreciation. Thus, the price of physical capital in units of the non-housing consumption good (the numeraire) is $P_{kt} = 1/\phi_t$. Note that the market value of the existing capital stock drops by a factor of $\frac{\phi_{t-1}}{\phi_t}$ when the relative price of new capital goods declines over time. In this case, the rate of economic depreciation $\delta_t^e = 1 - \frac{\phi_{t-1}}{\phi_t}(1-\delta_k)$ will exceed pure physical decay $\delta_k$ approximately by the rate at which the price of physical capital falls (cf. Cummins and Violante, 2002; Greenwood, Hercowitz, and Krusell, 1997). Consequently, the zero-profit condition (7) states that

$$r_{ft} = r_{kt} - \delta_t^e \tag{8}$$

where $r_{kt} = rr_{kt}/P_{k,t-1}$ denotes the gross return of an investment in physical capital, measured in terms of the numeraire.

## 2.4 Sequential Competitive Equilibrium

We assume an open economy with perfect international capital mobility. Thus, the domestic interest rate, $r_f$, is equal to the exogenous world interest rate and the interest rate sequence, $\{r_{ft}\}$, is exogenously given. Further, we consider a sequence of economic depreciation rates, $\{\delta_t^e\}$, as given, which will ultimately be pinned down by the exogenous sequence of productivity parameters, $\{\phi_t\}$. Finally, we take as given a sequence of measures, $\{\mu_{j0}\}$, over states, $(a_{j0}, x_{j0}, m_{j0}, h_{j0})$, of new-born households. Here $\mu_j$ defines the initial endowment of households born in period $j$. Note that this measure need not be a probability measure since some capital might be destroyed when passed on to a new generation and new-born households begin life with an initial endowment of human capital.

We next define the market clearing conditions. To this end, note that by a law of large numbers the value of any aggregate variable in period $t$ is defined by taking the expectations of the corresponding individual variable. For example, for the aggregate demand for housing stock we have $X_t = \sum_{j \leq t} E_t[x_{tj}]$, where $\sum_{j \leq t} E_t[x_{tj}]$ stands for the expectation of the variable $x_{jt}$ taken over individual histories, $(z_j^t, \eta_j^t, \epsilon_j^t)$, and individual initial states, $(a_{j0}, x_{j0}, m_{j0}, h_{j0})$. In equilibrium, the following three market clearing conditions have to hold:

$$
\begin{aligned}
H_t &= \sum_z E_t[h_{zt}]\pi_z \\
1 &= \sum_z E_t[x_{zt}]\pi_z \qquad (9) \\
1 &= \sum_z E_t[c_{x,zt}]\pi_z
\end{aligned}
$$

The first equation in (9) states that the human capital supplied by all households equals the human capital employed by the representative firm. The second condition says that the supply of the housing stock (land), which is normalized to one, is equal to the aggregate demand for housing equity by all households. The last equation states that the supply of housing consumption, which is also normalized to one, is equal to the aggregate demand for housing consumption by all households. Note that in this open-economy model there is no domestic market clearing condition for physical capital so that in general $P_{kt}K_t \neq \sum_{j \leq t} E_t[(a_{jt} + m_{jt})]$. Rather, there is international capital mobility at the world interest rate.

A sequential competitive equilibrium is defined as follows.

**Definition 1.** *Take as given an interest rate sequence, $\{r_{ft}\}$, a productivity sequence, $\{\phi_t, B_t\}$, and a sequence of initial distributions, $\{\mu_{j0}\}$. A sequential competitive equilibrium is a price sequence,$\{r_{mt}, rr_{kt}, rr_{ht}, rr_{xt}, P_t\}$ , and a family of household plans, $\{c_{jt}, c_{x,jt}, a_{jt}, x_{jt}, m_{jt}, h_{jt}\}$, satisfying*
*i) Utility maximization of households: Each plan $\{c_{jt}, c_{x,jt}, a_{jt}, x_{jt}, m_{jt}, h_{jt}\}$ maximizes expected lifetime utility with one-period utility function (1) subject to the household budget constraint (2).*
*ii) Profit maximization of production firms: In each period $t$, $(K_t, H_t)$ solves the first-order conditions (5).*
*iii) Profit maximization of financial intermediaries: In each period $t$, the no-arbitrage conditions (6) and (7) hold.*
*iv) Market clearing: The equalities (9) hold for all $t$.*

Suppose interest rate, economic depreciation rate, and initial distributions are constant over time. In this case definition 1 still applies and we can define a steady state equilibrium as a sequential equilibrium in which all endogenous aggregate variables are constant over time.

The definition of equilibrium does not include a government budget constraint. We assume that net government revenues are used for the consumption of a public good. For simplicity, we do not include consumption of the public good in the utility function (1).

# 3 Theoretical Results

In this section, we derive the main theoretical results. Proposition 1 characterizes the optimal decision rules of the household. Proposition 2 describes the sequential competitive equilibrium of the model economy.

## 3.1 Characterization of Household Problem

We next show that optimal consumption choices are linear in total wealth (human plus financial) and portfolio and default choices are independent of wealth. This property of the optimal policy function allows us to solve the model, which has considerable household heterogeneity and four inter-temporal choice variables $(a, h, x, m)$, without using approximation methods. The property also implies that the household decision problem is convex and the first-order approach can be utilized.

To state the characterization result, define the following variables:

$$
\begin{aligned}
& w_t = a_t + m_t + P_{t-1}x_t + h_t \\
& \theta_{at} = \frac{a_t}{w_t} \ , \ \theta_{ht} = \frac{h_t}{w_t} \ , \ \theta_{mt} = \frac{m_t}{w_t} \ , \ \theta_{xt} = \frac{P_{t-1}x_t}{w_t} \\
& \theta_t = (\theta_{at}, \theta_{ht}, \theta_{mt}, \theta_{xt}) \\
& r_{at} = (1-\tau)r_{ft} \\
& r_{ht} = (1-\tau)rr_{ht} - \delta_h(z_t) + \eta_t \\
& \tilde{r}_{mt} = (1-\tau)\begin{cases} r_{mt} & if\ d_t = 0 \\ -(1-\gamma) & if\ d_t = 1 \end{cases} \\
& \tilde{r}_{xt} = (1-\tau)\begin{cases} r_{xt} = \frac{(1+\epsilon_t)P_t + rr_{xt}}{P_{t-1}} - 1 & if\ d_t = 0 \\ -1 & if\ d_t = 1 \end{cases} \\
& r_t(\theta_t, d_t, s_t) = \theta_{at}r_{at} + \theta_{ht}r_{ht}(z_t, \eta_t) + \theta_{mt}\tilde{r}_{mt}(d_t) + \theta_{xt}\tilde{r}_{xt}(d_t, \epsilon_t)
\end{aligned}
$$

Here $s_t = (z_t, \epsilon_t, \eta_t)$ stands for the exogenous individual state, $w$ denotes the value of total wealth, $\theta$ is a vector of portfolio shares, and $r$ represents the corresponding total return on investment. Note that $w_t$ is total wealth before assets have paid off and depreciation has taken place, while $(1 + r_t(\theta_t, d_t, s_t))w_t$ is total wealth after asset payoff and depreciation has occurred. Specifically, the total return on the portfolio is a weighted average of the after-tax returns on risk-free financial assets, $r_a$, on risky human capital, $r_h$, and risky housing, $\tilde{r}_x$, as well as the mortgage rate, $\tilde{r}_m$, where the weights are the corresponding portfolio shares $\theta$.

Using the new variables, the budget constraint reads

$$
\begin{aligned}
w' &= (1+r(\theta,d,s))w - c - rr_x c_x \\
1 &= \theta'_a + \theta'_h + \theta'_m + \theta'_x \\
& \theta'_h \geq 0\,, \theta'_a \geq 0\,,\ \theta'_x \geq 0\,,\ \theta'_m \leq 0
\end{aligned}
\tag{10}
$$

Clearly, (10) is the budget constraint corresponding to an inter-temporal portfolio choice problem with linear investment opportunities and no exogenous source of income. However, in contrast to the standard portfolio choice problem, household make an additional default decision. Note also that the household problem depends on time through the effect of time-dependent interest rates, rental rates, and house prices on investment returns.

The representation of the household budget constraint shows that $(w,\theta,s)$ can be used as individual state variable and suggests the following solution to the household maximization problem. Consider the (time-dependent) Bellman equation associated with the household utility maximization problem

$$
\begin{aligned}
V_{yt}(w,\theta,s) &= \max_{c,c_x,w',\theta',d'} \left\{ \ln c + \nu \ln c_x + \beta \sum_{s'} \left[(1-\rho)V_{yt+1}(w',\theta',s') \ +\ \rho V_{ot+1}(w',\theta',s')\right] \pi(s') \right\} \\
V_{ot}(w,\theta,s) &= \max_{c,c_x,w',\theta',d'} \left\{ \ln c + \nu \ln c_x + \beta(1-q) \sum_{s'} V_{ot+1}(w',\theta',s')\pi(s') \right\} \\
& \textit{subject to} \\
w' &= (1+r(\theta,d,s))w - c - rr_{xt} c_x
\end{aligned}
\tag{11}
$$

where for simplicity we assume that the continuation value in the case of death is zero. We have the following characterization result for the solution to the household decision problem.

**Proposition 1.** *The optimal household plan is generated by a (in general time-dependent) household policy function. The household policy function and the associated value function (expected lifetime utility) for age group $z \in \{y,o\}$ are given by*

$$
\begin{aligned}
V_{zt}(w,\theta,s) &= \bar{V}_{zt} + \tilde{V}_z \ln\left((1+r(\theta,d(\theta,s),s))w\right) \\
c_z(w,\theta,s) &= \tilde{c}_z(1+r(\theta,d(\theta,s),s))w \\
c_{xzt}(w,\theta,s) &= \frac{\nu}{rr_{xt}} c_z(w,\theta,s) \\
w'_z(w,\theta,s) &= \left[1-(1+\nu)\tilde{c}_z\right](1+r(\theta,d(\theta,s),s))\,w
\end{aligned}
\tag{12}
$$

*where the optimal consumption-to-wealth ratios are* $\tilde{c}_z = 1/\tilde{V}_z$*, the value function coefficients are*

$$
\begin{aligned}
\tilde{V}_o &= \frac{1+\nu}{1-\beta(1-q)} \\
\tilde{V}_y &= \frac{1+\nu}{1-\beta(1-\rho)}\left[1+\frac{\beta\rho}{1-\beta(1-q)}\right] \\
\bar{V}_{ot} &= \nu[\ln\nu - \ln rr_{xt}] \\
&+ (1+\nu)\ln\tilde{c}_o + \beta(1-q)\tilde{V}_o\ln[1-(1+\nu)\tilde{c}_o] \\
&+ \beta(1-q)[\bar{V}_{ot+1} + EU_o] \\
\bar{V}_{yt} &= \nu[\ln\nu - \ln rr_{xt}] \\
&+ (1+\nu)\ln\tilde{c}_y + \beta\left[(1-\rho)\tilde{V}_y + \rho\tilde{V}_o\right]\ln[1-(1+\nu)\tilde{c}_y] \\
&+ \beta[(1-\rho)\bar{V}_{yt+1} + \rho\bar{V}_{ot+1} + EU_y]
\end{aligned} \tag{13}
$$

*the optimal portfolio choices,* $\theta_{y,t+1}$ *and* $\theta_{o,t+1}$*, are the solution to*

$$
\begin{aligned}
EU_y &= \max_{\theta_{y,t+1}}\sum_{\eta',\epsilon'}\left[(1-\rho)\tilde{V}_y\ln r_{y,t+1}(\theta_{y,t+1},d_{t+1},\eta',\epsilon') + \rho\tilde{V}_o\ln r_{o,t+1}(\theta_{y,t+1},d_{t+1},\eta',\epsilon')\right]\pi(\eta')\pi(\epsilon') \\
EU_o &= \max_{\theta_{o,t+1}}\sum_{\eta',\epsilon'}\tilde{V}_o\ln r_{o,t+1}(\theta_{o,t+1},d_{t+1},\eta',\epsilon')\pi(\eta')\pi(\epsilon')
\end{aligned} \tag{14}
$$

*and the optimal default decision follows a cut-off rule:*

$$
d_{t+1}(\epsilon_{t+1},\theta_{t+1}) = \begin{cases} 0 & if\ \epsilon_{t+1} \geq \epsilon^c_{t+1}(\theta_{t+1}) \\ 1 & otherwise \end{cases} \tag{15}
$$

*where the cut-off values* $\epsilon^c$ *are determined by the indifference condition:*

$$
\theta_{m,t+1}(1-\gamma+r_{m,t+1}) = \theta_{x,t+1}(1+r_{x,t+1}(\epsilon^c_{t+1}(\theta_{t+1}))) \tag{16}
$$

*Here* $r_y$ *and* $r_o$ *stand for the investment return when* $z_{t+1} = y$ *and* $z_{t+1} = o$*, respectively.*

*Proof*: See Appendix A.1.

The optimal default policy states that for every given portfolio state $\theta$ there exists a cut-off value $\epsilon^c$ for the house price shock such that if the realization of the shock is $\epsilon^c$ the household is indifferent between defaulting and repaying his debt. For better realizations of the house price shock, the household decides to repay; for worse, he defaults. Specifically, indifference condition (16) can be solved for the cut-off value as a function of the previously chosen loan-to-value ratio $-m_{t+1}/(P_t x_{t+1}) = -\theta_{m,t+1}/\theta_{x,t+1}$ :

$$\epsilon^c_{t+1}(\theta_{t+1}) = \frac{P_t}{P_{t+1}}\left[(1-\gamma+r_{m,t+1})\frac{-\theta_{m,t+1}}{\theta_{x,t+1}} - rr_{x,t+1}\right] - 1 \tag{17}$$

The maximization problem (14) is a convex problem so that first-order conditions are sufficient. Thus, to find the optimal portfolio we can confine attention to the first-order conditions. If the non-negativity constraints on portfolio choices are not binding, then the optimal portfolio choice for young households, $\theta_{y,t+1}$, is the solution to

$$\begin{aligned}
0 &= \sum_{\epsilon',\eta'}\left[(1-\rho)\tilde{V}_y\frac{r_{yh,t+1}(\eta')-r_{a,t+1}}{1+r_{y,t+1}(\theta_{y,t+1},d_{t+1},\epsilon',\eta')} + \rho\tilde{V}_o\frac{r_{oh,t+1}(\eta')-r_{a,t+1}}{1+r_{o,t+1}(\theta_{y,t+1},d_{t+1},\epsilon',\eta')}\right]\pi(\epsilon')\pi(\eta') \\
0 &= \sum_{\epsilon',\eta'}\left[(1-\rho)\tilde{V}_y\frac{\tilde{r}_{x,t+1}(d_{t+1},\epsilon')-r_{a,t+1}}{1+r_{y,t+1}(\theta_{y,t+1},d_{t+1},\epsilon',\eta')} + \rho\tilde{V}_o\frac{\tilde{r}_{x,t+1}(d_{t+1},\epsilon')-r_{a,t+1}}{1+r_{o,t+1}(\theta_{y,t+1},d_{t+1},\epsilon',\eta')}\right]\pi(\epsilon')\pi(\eta') \\
0 &= \sum_{\epsilon',\eta'}\left[(1-\rho)\tilde{V}_y\frac{\tilde{r}_{m,t+1}(d_{t+1})-r_{a,t+1}}{1+r_{y,t+1}(\theta_{y,t+1},d_{t+1},\epsilon',\eta')} + \rho\tilde{V}_o\frac{\tilde{r}_{m,t+1}(d_{t+1})-r_{a,t+1}}{1+r_{o,t+1}(\theta_{y,t+1},d_{t+1},\epsilon',\eta')}\right]\pi(\epsilon')\pi(\eta') \\
1 &= \theta_{ya,t+1}+\theta_{yh,t+1}+\theta_{yx,t+1}+\theta_{ym,t+1}
\end{aligned} \tag{18}$$

where the default choice, $d_{t+1}$, is determined by condition (15). Mutatis mutandis, the first-order conditions for the portfolio choice of old households, $\theta_{o,t+1}$, are the same.

Note that the equations in (18) state that marginal utility weighted expected excess returns on the various investment opportunities are equal to zero – a standard optimality condition in portfolio theories; the marginal utility of future consumption is equal to $\left(\frac{1}{\tilde{V}_z}(1+r_{t+1})w_{t+1}\right)^{-1}$ and therefore proportional to $\tilde{V}_z/(1+r_{t+1})$.

## 3.2 Characterization of Sequential Competitive Equilibria

Having defined sequential competitive equilibria in section 2.4, we now turn to the characterization of these equilibria. Specifically, we will express equilibrium prices as functions of the wealth distribution and derive the law of motion for the wealth distribution. Further, we will show that the relevant aggregate state variable is a 2-dimensional vector of aggregate total wealth held by the group of households aged $z \in \{y, o\}$.

Recall that the sequences $\{r_{ft}\}$, $\{\phi_t\}$, and $\{B_t\}$ are exogenous. Given these sequences, equation (7) determines the sequence $\{rr_{kt}\}$ and (5) then determines the sequences $\{\tilde{K}_t\}$ and $\{rr_{ht}\}$. To derive equilibrium prices, it will be convenient to define aggregate total wealth, including asset payoffs, that household group $z$ owns in period $t$ as: $\tilde{W}_{zt} = E[(1+r_t)w_t|z_t = z]\pi_z$. In the Appendix, we show that the household policy function implies that the market clearing conditions (9) reduce to

$$
(19) \qquad \begin{aligned}
H_{t+1} &= \theta_{yh,t+1}[1-(1+\nu)\tilde{c}_y]\tilde{W}_{y,t} \\
P_t &= \theta_{yx,t+1}[1-(1+\nu)\tilde{c}_y]\tilde{W}_{y,t} + \theta_{ox,t+1}[1-(1+\nu)\tilde{c}_o]\tilde{W}_{o,t} \\
rr_{xt} &= \nu[\tilde{c}_y\tilde{W}_{y,t} + \tilde{c}_o\tilde{W}_{o,t}]
\end{aligned}
$$

and the law of motion for total wealth by age group $(\tilde{W}_y, \tilde{W}_o)$ is given by

$$
(20) \qquad \begin{aligned}
\tilde{W}_{y,t+1} &= (1-\rho)\cdot[1-(1+\nu)\tilde{c}_y]\cdot(1+\bar{r}^{y|y}_{t+1})\cdot\tilde{W}_{y,t} \\
&+ \psi q\cdot[1-(1+\nu)\tilde{c}_o]\cdot(1+\bar{r}^{y|o}_{t+1})\cdot\tilde{W}_{o,t} + qW_e \\
\tilde{W}_{o,t+1} &= \rho\cdot[1-(1+\nu)\tilde{c}_y]\cdot(1+\bar{r}^{o|y}_{t+1})\cdot\tilde{W}_{y,t} \\
&+ (1-q)\cdot[1-(1+\nu)\tilde{c}_o]\cdot(1+\bar{r}^{o|o}_{t+1})\cdot\tilde{W}_{o,t}
\end{aligned}
$$

where $\bar{r}^{z'|z}_{t+1} = E[r(\theta_{t+1}, d(\theta_{t+1}, s_{t+1}), s_{t+1}|z', z]$ denotes the expected portfolio return conditional on age transition $(z', z)$, $\psi \leq 1$ is the fraction of wealth (financial capital) passed on from old households who died to new-born households, and $W_e$ is the initial endowment of human capital of new-born households.

**Proposition 2.** *Given an exogenous sequence $\{r_{ft}, \delta^e_t\}$ a sequential competitive equilibrium can be found as follows. The rental rates for physical and human capital, $\{rr_{kt}, rr_{ht}\}$, together with the sequence of capital-to-labor ratios, $\{\tilde{K}_t\}$, are determined by (7) and (5). The equilibrium sequence of mortgage rates, house prices, rental rates of housing, and wealth distribution, $\{r_{mt}, P_t, rr_{xt}, \tilde{W}_{yt}, \tilde{W}_{ot}\}$, together with the sequence of household portfolio and default choices, $\{\theta_t, d_t\}$, are the solution to the (6), (14), and (15), (19), and (20).*

Note that all household variables can be computed using the equilibrium portfolio choice, $\theta$, and the equilibrium law of motion for individual wealth, $w$, given in (12). Note further that all aggregate variables can be computed using the equilibrium portfolio choice, $\theta$, and the equilibrium law of motion for aggregate wealth (20). Thus, proposition 2 provides a complete characterization of sequential competitive equilibria.

A steady-state competitive equilibrium is a sequential competitive equilibrium with aggregate variables that are constant over time. Clearly, for a steady-state competitive equilibrium to exist, the exogenous variables $r_f$, $\delta^e_t$, and $W_e$ have to be constant over time. In the Appendix, we solve for the stationary wealth distribution. Given this stationary distribution, the market clearing conditions (19) and the mortgage pricing rule (6) determine steady state prices which in turn give rise to the steady state portfolio choice and aggregate variables.

Proposition 2 can be used to compute steady-state equilibria as follows. Take exogenous values $(r_f, \delta^e, W_e)$ as given and compute rental rates $rr_k$ and $rr_h$, together with the capital-labor ratio, $\tilde{K}$, using (7) and (5). To compute steady-state equilibria, solve the equations (6), (14), (15), (19), and (20) with $\tilde{W}_{y,t+1} = \tilde{W}_{y,t}$ and $\tilde{W}_{o,t+1} = \tilde{W}_{o,t}$.

Transitional dynamics can be computed by iterating over the sequence of total wealth by age group $(\tilde{W}_y, \tilde{W}_o)$, that is, over sequences of the relevant aggregate state variable. Specifically, take an exogenous sequence $\{r_{ft}, \delta_t^e\}$ that converges to a steady-state equilibrium as givenand compute this steady-state equilibrium as described above. Use equations (7) and (5) to compute a sequence of rental rates $\{rr_{kt}, rr_{ht}\}$, together with the sequence, $\{\tilde{K}_t\}$. Set the number of transitional periods $T$ large enough so that the economy converges to the steady-state equilibrium within $T$ periods. Guess a sequence of aggregate states, $\{(\tilde{W}_{y,t}, \tilde{W}_{o,t})\}_{t=0}^{T}$ , connecting initial and steady-state values of total wealth by age group. Begin the iteration over sequences of total wealth by age group as follows:

1. Given the sequences of exogenous variables, aggregate states and rental rates, start at period $T$ and solve backwards for a time series of individual households portfolio and default choices using the portfolio selection problem (14) and the default cut-off rule (15). To this end, note that the house price $P_{T+1}$ is the steady-state price and that the sequence of house prices can be derived sequentially backwards, employing the market clearing condition (19) given any guess for the sequence of total wealth by age group.

2. Given the time series for households' portfolio and default choices, start from the initial wealth distribution $(\tilde{W}_{y,0}, \tilde{W}_{o,0})$, use the corresponding law of motion for total wealth by age group (20), and solve forward for a new sequence of total wealth by age group.

3. Update the initial guess regarding the sequence of aggregate states, $\{(\tilde{W}_{y,t}, \tilde{W}_{o,t})\}_{t=0}^{T}$, by a weighted average of the initial and the implied sequence, and repeat this iteration with the update as guess until both sequences are sufficiently close to each other so that the update solves the transitional dynamics of total wealth by age group.

# 4 Calibration

The model economy is calibrated to match various stylized facts of the U.S. economy before real interest rates started to decline in 1985. In the following, we lay out our calibration strategy. We begin with parameters that are a priori known from the literature or are directly related to our targets and can be set immediately. Then, the remaining parameters are calibrated jointly by matching a set of targets. All parameters are listed in Tables 1 and 2.

**Direct Calibration. Demographics.** Let's begin with the demographic structure of the model population. We calibrate the ageing process to the following age groups: young households represent the working age population and old households retirees. Therefore the aging probability $\pi(o|y)$ and the death probability $\pi(y|o)$ are set to match an expected working life of 34 years (ages 25-59) and an expected duration of retirement of 30 years (ages 60-90), respectively.

**Taxation.** Next, consider the tax system. The income tax rate $\tau$ is set to 40%, as in Trostel (1993). This value matches the overall average marginal tax rate in the early 1980s well, as reported by Barro and Redlick (2011, Tab. 1) for the federal individual income tax, Social Security payroll tax, and state income taxes.

**Production Technology.** We follow Krebs, Kuhn, and Wright (2015) and set the parameter $\alpha$, which governs the capital elasticity of the Cobb-Douglas production function, to match a capital share in output of 32%.

The depreciation rate on human capital for old, i.e. retired, agents $\delta_h^o$ is set to 0.6. This means that agents lose 60 per cent of their human capital when they (stochastically) retire, while 40 per cent of their human capital is transformed into financial assets in order to capture pensions. The latter value corresponds to the social security replacement rate for a sixty-five-year-old medium-earnings worker in 1985, as reported by Diamond and Gruber (1999, Fig. 11.5). In addition, a depreciation rate of 60% is high enough to ensure that retired agents do not want to invest in human capital any more.

**Investment returns.** The real risk-free rate $r_f$ is calibrated to 4%, which is close to Huggett, Ventura, and Yaron (2011)'s value of 4.2% and the return on long-term, high-grade corporate bonds in the early 1980s, as reported by McGrattan and Prescott (2003).[1]

**Banks.** In the banking sector, the recovery rate in case of foreclosure $\kappa$ needs to be calibrated. For simplicity, we set $\kappa = 0$. That is, financial intermediaries need to write-off the mortgage claim completely.

**Endowments.** New-born households receive an initial endowment of human capital and a bequest which amounts to the fraction $\psi$ of total wealth of all households who die. For simplicity, we set $\psi = 0$. The initial endowment of newborns $w_e$ just scales the aggregate wealth level. Hence, we normalize $w_e = 1$.

**Risk.** Finally, individual house price risk needs to be calibrated. As Jeske, Krueger, and Mitman (2013, p. 926) have already recognized, the choice of the distribution function determines the rate of mortgage default. Therefore, the shape of the distribution of price shocks is especially important. In order to gain sufficient flexibility to match this default rate, we assume that the stochastic process of individual house price shocks consists of

[1] McGrattan and Prescott (2003, p. 395) argue that "high-grade bond returns are a good proxy for the returns that savers can realize on pension funds and annuities".

a normally distributed term that captures house price risk in normal times, $\epsilon^N$, and a random term, $\epsilon^D$, that picks up low-probability housing disasters: $\epsilon = (1+\epsilon^N)\cdot\epsilon^D - 1$.

If a disaster occurs, $\epsilon^D$ takes the value $\underline{\epsilon}^D \in [0,1]$; otherwise $\epsilon^D = 1$. The probability of a disaster $\pi_d$ is constant over time. The house price shock in normal times $\epsilon^N$ is calibrated to a mean of zero and a standard deviation of 15% which is in the middle of the empirical estimates.[2] In practice, we employ a two-state Gauss-Hermite approximation which yields the following two possible shock realizations: $-\epsilon_1^N, +\epsilon_1^N$.

We calibrate housing disasters to correspond to mortgage default. To ensure that the household defaults if and only if a housing disaster occurs, the default cut-off rule (15) requires that the disaster state $\underline{\epsilon}^D$ satisfies $(1+\epsilon^N)\cdot\underline{\epsilon}^D < 1+\epsilon^c \leq 1+\epsilon^N\ \forall\ \epsilon^N$ and, hence, in particular $\underline{\epsilon}^D < (1+\epsilon^c)/(1+\epsilon_1^N) \leq (1-\epsilon_1^N)/(1+\epsilon_1^N) = 0.74$. For simplicity, we set $\underline{\epsilon}^D = 0$. Then, the inequality constraint becomes $0 < \epsilon^c \leq 1-\epsilon_1^N = 0.85$ so that the cost of default $\gamma$ needs to satisfy

$$(21)\quad 1 + r_{m,t+1} - \frac{\theta_{x,t+1}}{-\theta_{m,t+1}} rr_{x,t+1} > \gamma \geq 1 + r_{m,t+1} - \frac{\theta_{x,t+1}}{-\theta_{m,t+1}}\left[rr_{x,t+1} + \frac{P_{t+1}}{P_t}(1-\epsilon_1^N)\right]$$

Note that for a loan-to-value ratio of 75% – which results from the aggregate targets matched in the indirect calibration subsection – the RHS of this constraint is likely to be negative, while the LHS is likely to be positive. Hence, we set $\gamma = 0$. Finally, as the realization of the disaster state will trigger mortgage default, we choose a probability $\pi_d$ of 1% which is in the range of values reported in Corbae and Quintin (2015); Jeske, Krueger, and Mitman (2013) as foreclosure rates and data on delinquency rates on single-family residential mortgages published by the Federal Reserve Board.

**Indirect Calibration.** Having selected the parameters that are directly related to our targets, we now turn to the parameters which are calibrated jointly by solving the model and matching a set of model statistics with their data equivalents.

**Preferences.** First, consider the preference parameters. As usual, we calibrate the discount factor $\beta$ to match the expected consumption growth rate of young households which was about 0% in the U.S. economy. Next, we choose the utility weight of housing services consumption $\nu$ so that the model matches a land value of 40% of GNP as can be computed from Davis and Heathcote (2007). The calibrated parameter value implies a rent-price ratio of 5.1% which is close to the values reported by Davis, Lehnert, and Martin (2008) for owner-occupied housing in the U.S. around 1980.

**Technology.** On the technology side, there are the depreciation rates and the total factor productivity to be calibrated. We use the depreciation rate on physical capital $\delta_k$ to target an saving rate of 22% which corresponds to U.S. data for the early 1980s.

[2] There are several estimates of the cross-sectional house price volatility in the literature (e.g. Campbell and Cocco, 2003, 2015; Corbae and Quintin, 2015; Glaeser, Gyourko, and Saiz, 2008; Zhou and Haurin, 2010), ranging from 11.5% to 22%.

**Table 1:** Direct Calibration

| Parameter | Symbol | Value | Source |
|---|---|---|---|
| *Technology* | | | |
| capital share | $\alpha$ | 0.32 | Krebs, Kuhn, and Wright (2015) |
| depreciation rate on $H$ for old | $\delta_h^o$ | 0.6 | Diamond and Gruber (1999) |
| risk-free rate | $r_f$ | 4% | |
| *Institutions* | | | |
| cost of default | $\gamma$ | 0 | ad hoc |
| recovery rate at foreclosure | $\kappa$ | 0 | ad hoc |
| income tax rate | $\tau$ | 0.4 | Barro and Redlick (2011); Trostel (1993) |
| initial endowment | $w_e$ | 1 | normalization |
| intergenerational transfer | $\psi$ | 0 | ad hoc |
| *Transition probabilities* | | | |
| default probability | $\pi_d$ | 0.01 | intermediate value |
| aging probability | $\pi(y\|y)$ | 33/34 | |
| survival probability | $\pi(o\|o)$ | 29/30 | |
| *Shocks* | | | |
| std. dev. of house price shock | $std(\epsilon)$ | 0.15 | intermediate value |

Regarding human capital, we set the depreciation rate for young $\delta_h^y$, i.e. working age, agents such that the model economy matches an investment rate of 21%.

Specifically, we consider a broad concept of human capital which includes education, on-the-job training as well as health. Snyder, de Brey, and Dillow (2018) report that total expenditures for education amount to 6% of GDP in 1985. Mincer (1998) points out that these expenditure figures do not include opportunity costs of students. He argues that at the post-secondary level these opportunity costs are on average about as high as the direct costs. According to Snyder, de Brey, and Dillow (2018), the latter amounted to more than 2% of GDP in 1985. Hence, total investments in education add up to at least 8% of GDP. Regarding the costs of job training in the U.S., Mincer (1991) provides estimates of worker and employer investments into on-the-job training. For 1987, his estimates range from \$240 billion to \$330 billion or 5 to 7 per cent of GDP. Therefore, we select the mid-point of 6% for job training investment. Finally, the National Health Expenditure Accounts, compiled by the Centers for Medicare & Medicaid Services, record national health expenditures of more than 10% of GDP for 1985; about two thirds of them were caused by people younger than 65.[3] Thus, we consider health expenditures of 7% of GDP as human capital investments. In sum, these three types of human capital investments add up to our target value of 21% of GDP.

[3]From 2002 through 2012, the National Health Expenditure Accounts provide biannual data on personal health care spending by age group. The combined share of all age groups younger than 65 is constant over time and fluctuates between 65.1% and 66.3%.

**Table 2:** Indirect Calibration

| Parameter | Symbol | Value | Target | Value | Source |
|---|---|---|---|---|---|
| *Preferences* | | | | | |
| utility weight of housing service | $\nu$ | 0.0541 | $p_x X/GNP$ | 40% | Davis and Heathcote (2007) |
| discount factor | $\beta$ | 0.9585 | $E[c'_y/c_y]-1$ | 0% | |
| *Technology* | | | | | |
| total factor productivity | $B$ | 0.2811 | $IIP/GNP$ | 0% | BEA |
| depreciation rate on $K$ | $\delta_k$ | 10.57% | $S/GNP$ | 22% | NIPA |
| depreciation rate on $H$ for young | $\delta_h^y$ | 3.30% | $I_H/GNP$ | 21% | |
| cost of financial intermediation | $\Delta$ | 0.010% | $p_m M/GNP$ | -30% | FoF |
| *Shocks* | | | | | |
| std. dev. of labor shock | $std(\eta)$ | 0.1246 | $\theta_h^y \cdot std(\eta)$ | 10.5% | Carroll (1992) |

In sum, this calibration strategy yields an aggregate physical-capital-to-GNP ratio of about 2.1 and a human-capital-to-GDP ratio of 4.2. Finally, the TFP parameter $B$ is used to bring the capital-human-capital ratio $\tilde{K}$ which is indirectly determined by our target value for the IIP-GNP ratio of 0 in line with the implied values for the rental rates of physical capital and human capital.

**Banks.** In the banking sector, there is one more parameter to be calibrated: the cost of financial intermediation $\Delta$. Since empirical estimates vary considerably, ranging from 0.11% to 2.18% (Mehra, Piguillem, and Prescott, 2011; Mitman, 2016; Philippon, 2012), we calibrate this parameter to match the ratio of mortgage debt to GNP which is 30% in the early 80s.

**Risk.** Finally, the labor shock has to be calibrated. We choose the labor shock $\eta$ to be normally distributed with zero mean. Its standard deviation is set to match observed labor income risk $\theta_h^y \cdot \sigma_\eta$ of 10.5%, as reported by Carroll (1992). This requires a standard deviation of 12.5%. In our quantitative analysis we consider a five-state approximation of the labor shock that is based on Gauss-Hermite quadrature.

# 5 Quantitative Analysis

In this section, we use the calibrated model economy to simulate the consequences of the long-run decline in real interest rates for the U.S. economy. We next present the settings of our computational experiment which mimics the long-run decline in real interest rates observed since the early 1980s. Then we discuss the results of our quantitative analysis of the model economy, beginning with a discussion of the steady-state effects. Finally, we illustrate the macroeconomic effects during the transition period, comparing

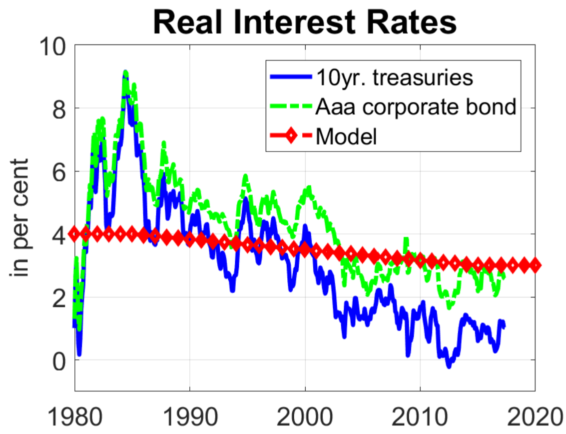


**Figure 1:** Long-Run Decline in Real Interest Rates

NOTES: Real interest rates are computed as ex-ante expected rates using one-year-ahead inflation expectations. These expectations are generated as HP-filtered trend in CPI inflation rates (Oliveira Martins, Scarpetta, and Pilat, 1996). For the HP-filtering of monthly data, a lambda value of 129600 is used (Ravn and Uhlig, 2002). DATA SOURCES: FRED, Federal Reserve Bank of St. Louis.

our simulation results to actual U.S. data for the last four decades.

## 5.1 Computational Experiment

The computational experiment intends to capture the long-run decline in real interest rates which the U.S. economy has experienced since the early 1980s. Figure 1 illustrates this declining trend with the help of real returns on 10-year treasuries and high-grade corporate bonds. At the same time, however, the real return on business capital, as recorded in the NIPAs, did not decline. Rather, it stayed more or less constant during the post-war period at a level of 4% to 5% (Caballero, Farhi, and Gourinchas, 2017; Gomme, Ravikumar, and Rupert, 2011; McGrattan and Prescott, 2003). This means that the spread between financial returns and the measured returns to productive capital started to rise in the early 1980s.

One explanation for this divergence in measured returns might be an acceleration in the decline of constant-quality investment good prices which also occurred in the early 1980s (Cummins and Violante, 2002; Pakko, 2005). According to Greenwood, Hercowitz, and Krusell (1997) and others, the NIPAs do not reflect the full scope of quality improvements in investment goods. Consequently, the NIPAs tend to underestimate economic depreciation due to obsolescence. Hence, faster obsolescence that leads to losses in the value of capital assets which are underestimated in the NIPAs is consistent with the observation that a gap between financial returns and the measured returns to capital opened up.

At this point, we do not want to embrace this explanation. Rather, we just take the observation that the return to capital stayed constant, while interest rates declined as given when conducting our computational experiment. Specifically, we consider a gradual (linear) decline in the risk-free rate from 4% in 1985 to 3% in 2015 and thereafter, as depicted in Figure 1 by the red line. A glance at Figure 1 might suggest that the 1%-decline in interest rates we consider might be a bit too small.[4] However, we believe that there are good reasons to be conservative. Firstly, real interest rates in the early 1980s fluctuate a lot due to the Volcker disinflation with negative values in 1979 and almost two-digit values five years later. Secondly, the model presumes that households can immediately and costlessly adjust their portfolios, while in the real world their reaction might be more sluggish. From this view, a conservative change in the risk-free rate that dampens portfolio reallocations is desirable. Further, we assume that the rental rate of physical capital remains constant forever at 4% so that a gap between this return on physical capital and the investor's risk-free rate gradually opens as in the data.[5]

Finally, we need to endow the households in our model economy with expectations about the future path of interest rates. To this end, we closely follow the "shocks to expectations" information structure by Garriga, Manuelli, and Peralta-Alva (2018). They introduce shocks to expectations or surprises as follows: Initially, the economy is in a perfect-foresight steady state and households have some initial expectations about the interest rate path set by the initial steady-state risk-free rate $r_f^*$ that is expected to remain unchanged in the future. In our case, $\{r_{f,t}^e = 0.04\}_{1985}^{\infty}$. Then the risk-free rate suddenly declines and households are surprised by this decline. Therefore, households adjust their expectations. By assumption, households expect the new prevailing interest rate as permanent. In each subsequent period households are surprised by an unanticipated interest-rate shock and, again, perceive the new risk-free rate as permanent. Finally, from 2015 onwards, the interest rate remains constant at 3% and households' expectations, $\{r_{f,t}^e = 0.03\}_{2015}^{\infty}$, are in line with perfect foresight again.

In sum, our computational experiment involves time pathes for the risk-free rate, $\{r_{f,t}\}_{1985}^{\infty}$, and the rental rate of physical capital, $\{rr_{k,t}\}_{1985}^{\infty}$, as well as a set of interest-rate path expectations, $\{\{r_{f,\tau}^e = r_{f,t}\}_t^{\infty}\}_{1985}^{\infty}$.

## 5.2 Quantitative Results

In this section, we use the calibrated model economy to simulate the consequences of the long-run decline in real interest rates. We begin with an analysis of the long-run effects on some of the main macroeconomic variables. Then we assess the ability of the model to reproduce actual time-series data for U.S. economy during the last four decades.

[4] Kiyotaki, Michaelides, and Nikolov (2011) also view a fall in the world real interest rate by 1 percentage point as reasonable change to fundamentals in the U.S. economy and study its impact on housing markets and macroeconomic aggregates.

[5] This is in line with Garriga, Manuelli, and Peralta-Alva (2018).

**Table 3:** Long-Run Macroeconomic Effects of the Decline in Real Interest Rates

| | Variable | Model | | |
|---|---|---|---|---|
| | | Pre | Post | Change |
| **Risk-free interest rate** | $r_f$ | 4.00% | 3.00% | -25.0% |
| **Rental rate of $H$** | $rr_h$ | 15.23% | 15.23% | 0.0% |
| **Mortgage rate** | $r_m$ | 5.06% | 4.05% | -20.0% |
| **Rent - price ratio** | $r_x$ | 5.14% | 4.15% | -19.3% |
| **Rent** | $rr_x$ | 0.0044 | 0.0050 | +13.6% |
| **Price of land** | $P_x$ | 0.0857 | 0.1200 | +40.0% |
| **Gross national product** | $GNP$ | 0.2144 | 0.2448 | +14.2% |
| **Human capital - GNP ratio** | $H/GNP$ | 4.23 | 4.94 | +16.8% |
| **Phys. capital - GNP ratio** | $K/GNP$ | 2.08 | 2.43 | +16.8% |
| **Land - GNP ratio** | $PX/GNP$ | 40.0% | 49.0% | +22.5% |
| **Mortgage debt - GNP ratio** | $-M/GNP$ | 30.0% | 39.1% | +30.3% |
| **Wealth - GNP ratio** | $W/GNP$ | 6.71 | 6.68 | -0.4% |
| **Saving - GNP ratio** | $S/GNP$ | 22.0% | 13.2% | -40.0% |
| **Phys. investment - GNP ratio** | $I_K/GNP$ | 22.0% | 25.4% | +15.5% |
| **Educ. investment - GNP ratio** | $I_H/GNP$ | 21.0% | 24.5% | +16.7% |
| **BoP - GNP ratio** | $(S-I_K)/GNP$ | 0.0% | -12.2% | |

**Long-Run Macroeconomic Effects.** Table 3 presents the long-run effects of the decline in real interest rates, where the long-run effects are computed by comparing the steady-state values before the decline with the steady-state values after the decline. Table 3 reveals that the equilibrium mortgage rate and rent-price ratio decline together with the risk-free rate, while the rental rate of human capital remains constant by construction. Hence, after the interest-rate decline human capital is a more attractive investment so that both the share of education investment in GNP and the human-capital-GNP ratio increase by almost 17%. In an open economy, the increase in human capital results in a proportionate increase in physical capital, causing considerable capital inflows. This leads to an increase in GNP by 14%.

Besides, the decline in interest rates fosters the demand for land, pushing its price by 40%. The additional housing demand of 9% of GNP is entirely financed by mortgage debt so that the aggregate loan-to-value ratio increases from 75% to almost 80% (= 39.0% / 48.9%). Since the rent only rises by 13% – which is more or less in line with GNP – the rent-price ratio declines by 19% to a value of 4.15% in the new steady state.

**Transitional Dynamics.** Now we assess the ability of the model to reproduce several important developments of U.S. economy during the last four decades. To this end, we compare the simulation results of our computational experiment to actual U.S. time-series data. We begin with a discussion of the implications of an interest-rate decline for individual household behavior.

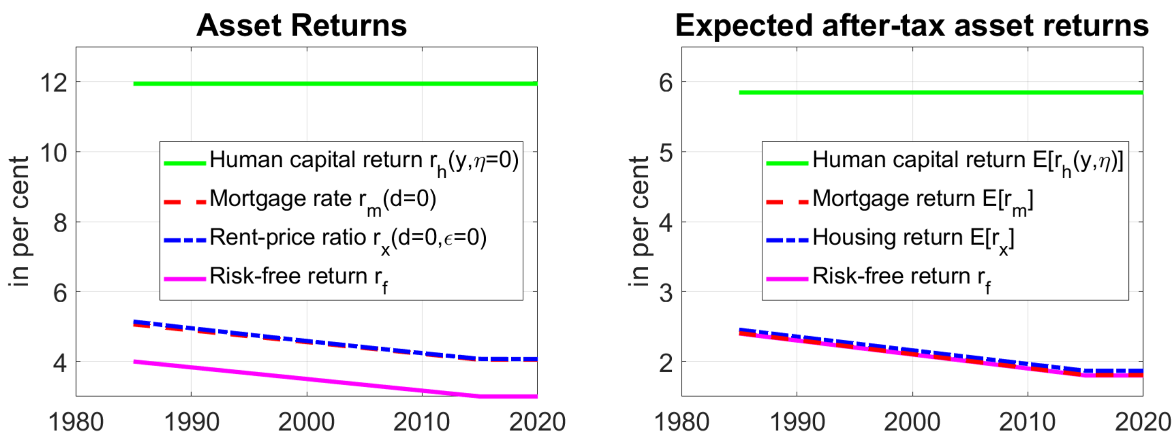


**Figure 2:** Investment Returns

**Effects on household behavior.** Figure 2 demonstrates the effects of a gradual one-percent decline in real interest rates on investment returns during the transition period. We see that with the risk-free rate the equilibrium values of the mortgage rate and the rent-price ratio also gradually decline so that their excess returns hardly change. From the mortgage pricing rule (6) it clear that under the current calibration any change in the risk-free rate is passed on to the mortgage rate with only marginal amplification.[6] For the housing return, the intuition derives from market clearing. Since the housing stock is in fixed supply, market clearing in this market means that households need to be willing to hold exactly this quantity and the house price has to adjust correspondingly. Suppose housing was the only asset in this economy, as in the Lucas (1978) exchange economy. Then market clearing would require the households' housing portfolio share to equal 1. In particular, it would not change in equilibrium due to a decline in interest rates. Suppose further that Merton (1969) conditions would apply so that the solution of the households' portfolio choice problem depends only on the excess return of housing and its variance which is exogenously given. Then, the decline in the risk-free rate needs to be accompanied by an equi-sized decline in the housing return in order to keep the excess return and, hence, the portfolio share unaffected. It appears that in our framework with four almost uncorrelated assets[7] much of the preceding reasoning remains valid.

Figure 3 illustrates how these trends in investment returns affect portfolio choices of households. At first glance we recognise a significant portfolio shift from the risk-free asset into risky human capital for young households. Specifically, between 1985 and 2020 the model predicts an increase in the human capital share from 84.3% to 97.2%, while at the same time the risk-free share declines from 14.3% to 1.4%. The key driver for this significant reallocation is an increase in the excess return of human capital by 1%.

[6]Totally differentiating the mortgage pricing rule (6) yields the equilibrium change in the mortgage rate as $dr_m = 1/\left(1 - \pi_d(1-\kappa)\right) dr_f$, with $\pi_d = 0.01$ and $\kappa = 0$ under the current calibration.

[7]There is only correlation between housing and mortgage return through default.

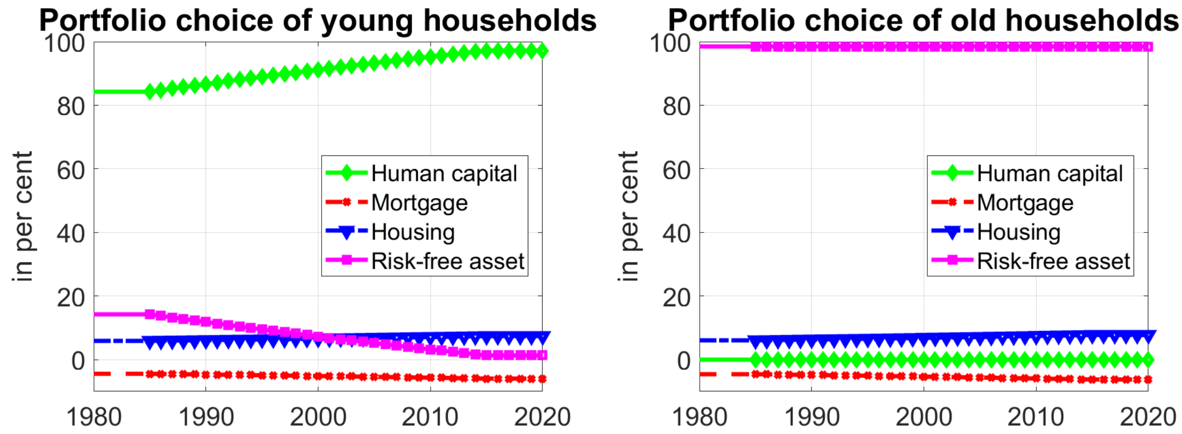


**Figure 3:** Portfolio Choices

Besides, young households further increase their risk exposure by borrowing additional 1.5% of their wealth and investing receipts in housing. As a result, the housing share increases from 5.9% to 7.4%, mortgage debt from 4.4% to 5.9% and leverage from 75.0% to 80.2%.

The portfolio choice of old households reacts less to the decline in interest rates. The main reason is that old households do not hold human capital any longer and the other investment returns decline almost in parallel with the risk-free rate. Therefore, the portfolio share of the risk-free asset remains constant at 98.5%. However, the decline in investment returns induces old households to take on a bit more risk. Specifically, they increase their housing portfolio share from 6.1% to 7.8% and finance the additional housing entirely by mortgage debt. This increases the loan-to-value ratio from 75.0% to 80.2%.

An immediate consequence of young households' portfolio response to the decline in interest rates is that their exposure to labor market risk goes up. Figure 4 displays how this portfolio shift towards human capital translates into higher cross-sectional inequality in labor incomes $\theta_y r_h(y, \eta_i) w_i$. Overall, the model can account for a little more than half of the increase in the variance of log labor income between 1985 and 2005, as reported by Heathcote, Perri, and Violante (2010). In addition, our computational experiment matches the growing trend in the labor income share of the top decile, documented by Piketty, Saez, and Zucman (2018), quite well.

**Macroeconomic implications.** Figure 5 shows the transitional dynamics of several macroeconomic variables after the gradual decline in interest rates and compares them to their actual empirical counterparts. By and large, the computational experiment we consider can account for most of the important developments of U.S. economy during the last four decades. Specifically, the model predicts recent trends in the U.S. housing market very well. The long-run increase in both the price-rent ratio and the market

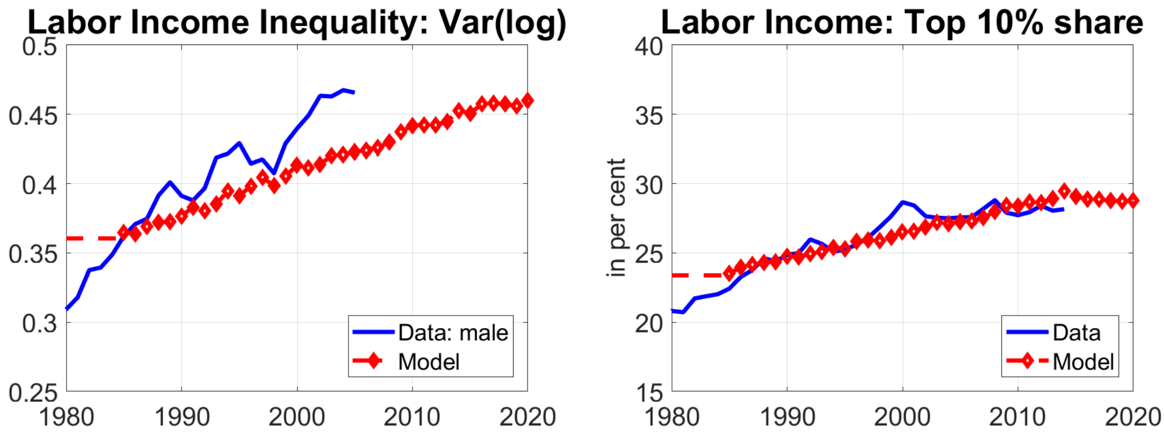


**Figure 4:** Labor Income Inequality

DATA SOURCES: Heathcote, Perri, and Violante (2010); Piketty, Saez, and Zucman (2018).

value of housing are entirely captured as well as about half of the boom in mortgage debt. Obviously, the model is silent about the boom-bust cycle of the 2007 financial crisis which the literature partly attributes to a relaxation of lending standards or adaptive expectations.

In the capital market, the experiment is able to quantitatively explain the declining trend in national savings, while its performance regarding investment is a bit worse. In particular, the model cannot account for the drop in investment concomitant with the financial crisis which should not be surprising either. As a consequence, the simulations overestimate actual capital inflows resulting in a net international investment position that is worse than in U.S. data. When assessing the model's performance in this dimension, one should, however, keep in mind that U.S. households and businesses hold a sizable stock of foreign assets and receive positive interest income from the rest of the world despite their net debtor investment position. These observations are clearly beyond the scope of our model.

Finally, the computational experiment also predicts a significant rise in human capital investments following from declining interest rates and, hence, rising excess returns. More precisely, during the 30 years of declining real rates, human capital investments temporarily hike up between 10% and 35%, while in the remaining years of the transition period the investment rate is about 20% higher than its 1985 level. By and large, the model simulations capture the time-series patterns of U.S. educational expenditures quite well, just predicting slightly higher human capital investment rates, especially during the first phase of the transition period. However, note that our empirical measure confines itself to educational expenditures, while our model is calibrated to a broad measure of human capital including on-the-job training and health expenditures. In particular, the latter were quickly rising in the U.S. during the last four decades.

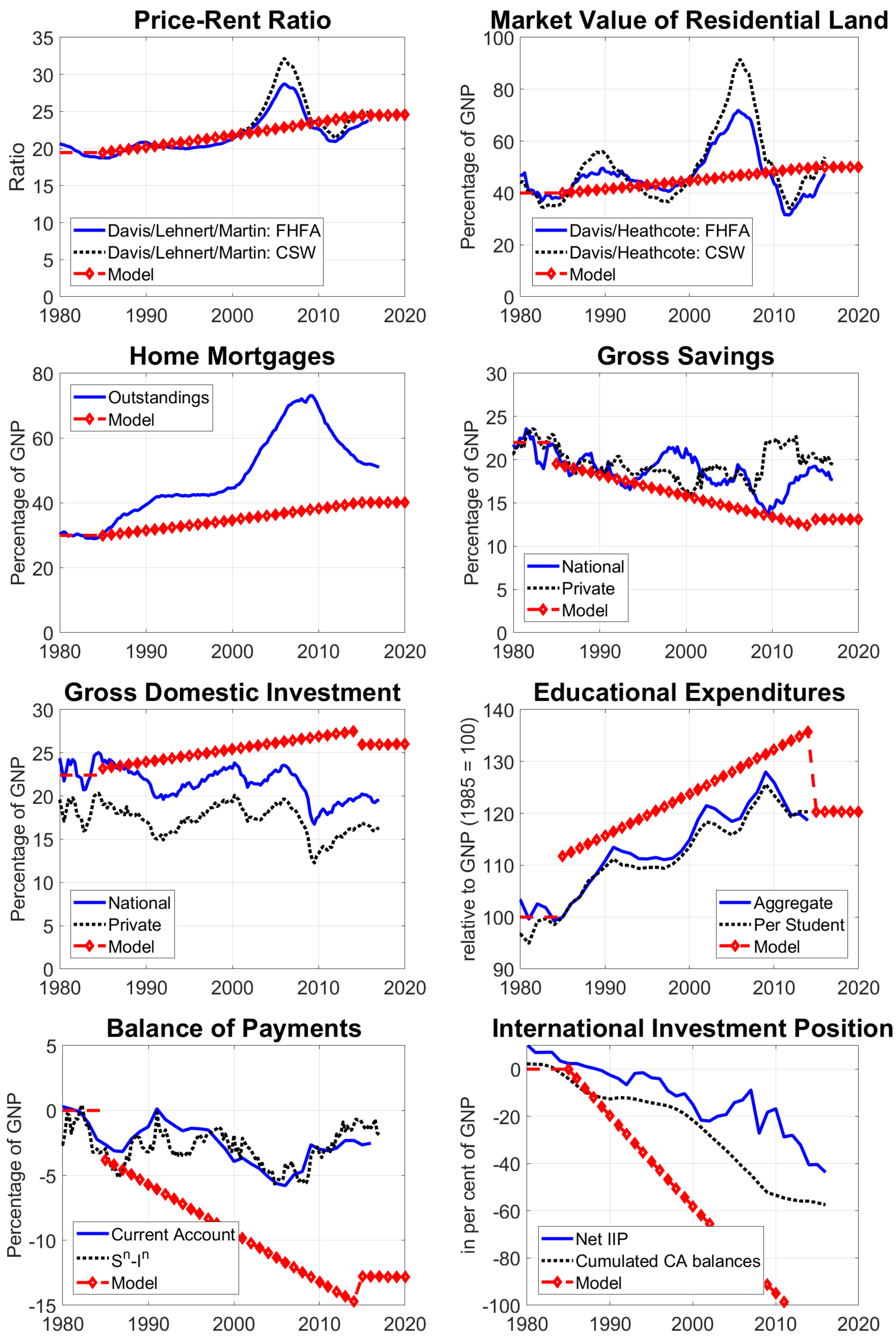


**Figure 5:** Transitional Macroeconomic Effects of the Decline in Real Interest Rates
NOTES: Data for educational expenditures only covers total expenditures for education, as reported by Snyder, de Brey, and Dillow (2018), and does neither include opportunity costs of students, nor on-the-job training, nor health expenditures.
DATA SOURCES: FRED, Federal Reserve Bank of St. Louis; Davis, Lehnert, and Martin (2008)/Davis and Heathcote (2007)/Lincoln Institute of Land Policy; Snyder, de Brey, and Dillow (2018).

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# A Model Appendix

## A.1 Proof of proposition 1: Solution to the Bellman equation

We solve the Bellman equation of the households decision problem (11) by the guess-and-verify method. Our guess for age group $z \in \{y, o\}$ is

$$
\begin{aligned}
V_{zt}(w,\theta,s) &= \bar{V}_{zt} + \tilde{V}_z \ln\left((1+r(\theta,d(\theta,s),s))w\right) \\
c_z(w,\theta,s) &= \tilde{c}_z(1+r(\theta,d(\theta,s),s))w \\
c_{xzt}(w,\theta,s) &= \frac{\nu}{rr_{xt}} c_z(w,\theta,s) \\
w'_z(w,\theta,s) &= \left[1-(1+\nu)\tilde{c}_z\right](1+r(\theta,d(\theta,s),s))\,w
\end{aligned} \tag{22}
$$

First, consider the decision problem of old households ($z = o$) who will die with constant probability $q$. Substituting this guess into the Bellman equation yields

$$
\begin{aligned}
& \bar{V}_{ot} + \tilde{V}_o \ln\left((1+r(\theta,d(\theta,s),s))w\right) \\
= \; & \max_{\tilde{c}_o,\theta',d'} \Big\{ (1+\nu)\left[\ln\tilde{c}_o + \ln\left((1+r(\theta,d(\theta,s),s))w\right)\right] + \nu\ln\nu - \nu\ln rr_{xt} \\
+ \; & \beta(1-q)\left[\bar{V}_{ot+1} + \tilde{V}_o \sum_{s'} \ln\left((1+r(\theta',d',s'))w'\right)\pi(s')\right]\Big\} \\
= \; & (1+\nu)\ln\left((1+r(\theta,d(\theta,s),s))w\right) + \nu\ln\nu - \nu\ln rr_{xt} \\
+ \; & \beta(1-q)\bar{V}_{ot+1} \\
+ \; & \beta(1-q)\max_{\theta',d'}\left\{\tilde{V}_o \sum_{s'} \ln(1+r(\theta',d',s'))\pi(s')\right\} \\
+ \; & \max_{\tilde{c}_o}\Big\{(1+\nu)\ln\tilde{c}_o + \beta(1-q)\tilde{V}_o\ln[1-(1+\nu)\tilde{c}_o]\Big\} \\
+ \; & \beta(1-q)\tilde{V}_o \ln\left((1+r(\theta,d(\theta,s),s))w\right)
\end{aligned} \tag{23}
$$

The guess works for

$$
\begin{aligned}
\tilde{V}_o &= \frac{1+\nu}{1-\beta(1-q)} \\
\tilde{c}_o &= \frac{1}{\tilde{V}_o} = \frac{1-\beta(1-q)}{(1+\nu)} \\
\bar{V}_{ot} &= \nu[\ln\nu - \ln rr_{xt}] \\
&+ (1+\nu)\ln\tilde{c}_o + \beta(1-q)\tilde{V}_o\ln[1-(1+\nu)\tilde{c}_o] \\
&+ \beta(1-q)\bar{V}_{ot+1} \\
&+ \beta(1-q)\max_{\theta',d'}\left\{\tilde{V}_o\sum_{s'}\ln(1+r(\theta',d',s'))\pi(s')\right\}
\end{aligned} \tag{24}
$$

Now, consider the decision problem of young households ($z = y$) who will retire with constant probability $\rho$. Substituting our guess into their Bellman equation yields

$$
\begin{aligned}
& \bar{V}_{yt} + \tilde{V}_y \ln\left((1 + r(\theta, d(\theta, s), s))w\right) \\
= \; & \max_{\tilde{c}_y, \theta', d'} \{(1+\nu)\left[\ln \tilde{c}_y + \ln\left((1 + r(\theta, d(\theta, s), s))w\right)\right] + \nu \ln \nu - \nu \ln rr_{xt} \\
+ \; & \beta(1-\rho)\left[\bar{V}_{yt+1} + \tilde{V}_y \sum_{s'} \ln\left((1 + r(\theta', d', s'))w'\right)\pi(s')\right] \\
+ \; & \beta\rho\left[\bar{V}_{ot+1} + \tilde{V}_o \sum_{s'} \ln\left((1 + r(\theta', d', s'))w'\right)\pi(s')\right]\Bigg\} \\
= \; & (1+\nu)\ln\left((1 + r(\theta, d(\theta, s), s))w\right) + \nu \ln \nu - \nu \ln rr_{xt} \\
+ \; & \beta\left[(1-\rho)\bar{V}_{yt+1} + \rho \bar{V}_{ot+1}\right] \\
+ \; & \beta \max_{\theta', d'}\left\{\left[(1-\rho)\tilde{V}_y \sum_{s'} \ln(1 + r(\theta', d', s')) + \rho \tilde{V}_o \sum_{s'} \ln(1 + r(\theta', d', s'))\right]\pi(s')\right\} \\
+ \; & \max_{\tilde{c}_y}\left\{(1+\nu)\ln \tilde{c}_y + \beta\left[(1-\rho)\tilde{V}_y + \rho\tilde{V}_o\right]\ln[1 - (1+\nu)\tilde{c}_y]\right\} \\
+ \; & \beta\left[(1-\rho)\tilde{V}_y + \rho\tilde{V}_o\right]\ln\left((1 + r(\theta, d(\theta, s), s))w\right)
\end{aligned} \tag{25}
$$

The guess works for

$$
\begin{aligned}
\tilde{V}_y &= \frac{(1+\nu) + \beta\rho\tilde{V}_o}{1 - \beta(1-\rho)} = \frac{1+\nu}{1-\beta(1-\rho)}\left[1 + \frac{\beta\rho}{1-\beta(1-q)}\right] \\
\tilde{c}_y &= \frac{1}{\tilde{V}_y} = \frac{1-\beta(1-\rho)}{(1+\nu)} \frac{1-\beta(1-q)}{1-\beta(1-q)+\beta\rho} \\
\bar{V}_{yt} &= \nu[\ln \nu - \ln rr_{xt}] \\
&+ (1+\nu)\ln \tilde{c}_y + \beta\left[(1-\rho)\tilde{V}_y + \rho\tilde{V}_o\right]\ln[1-(1+\nu)\tilde{c}_y] \\
&+ \beta[(1-\rho)\bar{V}_{yt+1} + \rho\bar{V}_{ot+1}] \\
&+ \beta \max_{\theta', d'}\left\{\left[(1-\rho)\tilde{V}_y \sum_{s'} \ln(1 + r(\theta', d', s')) + \rho\tilde{V}_o \sum_{s'} \ln(1 + r(\theta', d', s'))\right]\pi(s')\right\}
\end{aligned} \tag{26}
$$

## A.2 Characterization of Sequential Competitive Equilibria

### A.2.1 Market Clearing Conditions

In the following, we use the household policy functions (12) to rewrite the market clearing conditions (9) and express equilibrium prices as functions of the wealth distribution. Let $\tilde{w}_{zt} = (1 + r_t)w_{zt}$ be the wealth of a household aged $z$ after all assets have paid off. The aggregate stock of human capital is

$$
\begin{aligned}
H_{t+1} &= \sum_z E_t[h_{z,t+1}]\pi_z \\
&= \sum_z E_t[\theta_{hz,t+1} w_{z,t+1}]\pi_z \\
&= \sum_z E_t[\theta_{hz,t+1}[1-(1+\nu)\tilde{c}_z](1+r_t)w_{zt}]\pi_z \qquad (27)\\
&= \sum_z \theta_{hz,t+1}[1-(1+\nu)\tilde{c}_z]E_t[\tilde{w}_{zt}]\pi_z \\
&= \theta_{hy,t+1}[1-(1+\nu)\tilde{c}_y]\tilde{W}_{y,t}
\end{aligned}
$$

where $\tilde{W}_{zt} = E[\tilde{w}_{zt}]\pi_z = E[(1+r_t)w_t|z_t = z]\pi_z$ denotes aggregate total wealth after assets have paid off that age group $z$ owns. The second line in (27) uses the definition of portfolio shares $\theta$. The third line uses the equilibrium law of motion for the individual wealth. The fourth line follows from the definition of total wealth after all assets have paid off and from the fact that the portfolio choices only depend on age $z$. The last line makes use of the definition of aggregate total wealth by age group and takes into account that old households do not invest in human capital , i.e. $\theta_{ho} = 0$. Similarly, the house price can be derived

$$
\begin{aligned}
1 &= \sum_z E_t[x_{z,t+1}]\pi_z \\
&= \sum_z E_t[\theta_{xz,t+1} w_{z,t+1}/P_t]\pi_z \\
P_t &= \sum_z E_t[\theta_{xz,t+1}[1-(1+\nu)\tilde{c}_z](1+r_t)w_{zt}]\pi_z \qquad (28)\\
&= \sum_z \theta_{xz,t+1}[1-(1+\nu)\tilde{c}_z]E_t[\tilde{w}_{zt}]\pi_z \\
&= \theta_{xy,t+1}[1-(1+\nu)\tilde{c}_y]\tilde{W}_{y,t} + \theta_{xo,t+1}[1-(1+\nu)\tilde{c}_o]\tilde{W}_{o,t}
\end{aligned}
$$

Finally, the rental rate of housing is derived using the household's consumption policy (in the second line of (29)

$$
\begin{aligned}
1 &= \sum_z E_t[c_{x,zt}]\pi_z \\
&= \sum_z E_t[\nu\tilde{c}_z(1+r_t)w_{zt}/rr_{xt}]\pi_z \\
rr_{xt} &= \nu \sum_z \tilde{c}_z E_t[\tilde{w}_{zt}]\pi_z \qquad (29)\\
&= \nu[\tilde{c}_y \tilde{W}_{y,t} + \tilde{c}_o \tilde{W}_{o,t}]
\end{aligned}
$$

Note that equations (27), (28), and (29) express tomorrow's stock of human capital as well as the current price and rental rate of housing as functions of the current aggregate state variables $(\tilde{W}_y, \tilde{W}_o)$ and, where applicable, current portfolio choices $\theta'$.

### A.2.2 Wealth Distribution

The law of motion for aggregate total wealth by age group is determined by the evolution of individual wealth. Individual wealth for those households that survive evolves according to the household's savings function, i.e. the last line of (12). In terms of total wealth after assets have paid off, this law of motion becomes

$$\tilde{w}_{t+1} = [1-(1+\nu)\tilde{c}_z]\,(1+r(\theta_{t+1}, d(\theta_{t+1}, s_{t+1}), s_{t+1}))\,\tilde{w}_t \tag{30}$$

Old households who die bequeath a fraction $\psi \leq 1$ of their wealth to new-born households. Besides, newborns are endowed with $w_e$. Consequently, aggregate total wealth of tomorrow's old households consists of the savings of currently young households who age and the savings of those old households who survive

$$\begin{aligned}
\tilde{W}_{o,t+1} &= E[\tilde{w}_{o,t+1}]\cdot\pi_o \\
&= E[\tilde{w}_{t+1}|o,y]\cdot\rho\pi_y + E[\tilde{w}_{t+1}|o,o]\cdot(1-q)\pi_o \\
&= E[[1-(1+\nu)\tilde{c}_z](1+r_{t+1})\tilde{w}_t|o,y]\cdot\rho\pi_y \\
&+ E[[1-(1+\nu)\tilde{c}_z](1+r_{t+1})\tilde{w}_t|o,o]\cdot(1-q)\pi_o \\
&= \rho\cdot[1-(1+\nu)\tilde{c}_y]\cdot(1+\bar{r}^{o|y}_{t+1})\cdot E[\tilde{w}_{yt}]\pi_y \\
&+ (1-q)\cdot[1-(1+\nu)\tilde{c}_o]\cdot(1+\bar{r}^{o|o}_{t+1})\cdot E[\tilde{w}_{ot}]\pi_o \\
&= \rho\cdot[1-(1+\nu)\tilde{c}_y]\cdot(1+\bar{r}^{o|y}_{t+1})\cdot\tilde{W}_{y,t} \\
&+ (1-q)\cdot[1-(1+\nu)\tilde{c}_o]\cdot(1+\bar{r}^{o|o}_{t+1})\cdot\tilde{W}_{o,t}
\end{aligned} \tag{31}$$

where $\bar{r}^{z'|z}_{t+1} = E[r(\theta_{t+1}, d(\theta_{t+1}, s_{t+1}), s_{t+1}|z', z]$ denotes the expected portfolio return conditional on age transition $(z', z)$. In (31) the second equality uses the law of iterated expectations, the third equality makes use of the equilibrium law of motion for $\tilde{w}$, the fourth equality follows from the fact that portfolio choices only depend on age $z$ in conjunction with the definition of $\bar{r}$, and the last equality is a direct implication of the definition of $\tilde{W}$. Similarly, aggregate total wealth of tomorrow's young households consists of the savings of currently young households who do not age, bequests from dying old households, and the initial (human capital) endowment of newborns

$$
\begin{aligned}
\tilde{W}_{y,t+1} &= E[\tilde{w}_{y,t+1}]\pi_y \\
&= E[\tilde{w}_{t+1}|y,y] \cdot (1-\rho)\pi_y + \psi E[\tilde{w}_{t+1}|y,o] \cdot q\pi_o + w_e q\pi_o \\
&= E[[1-(1+\nu)\tilde{c}_z](1+r_{t+1})\tilde{w}_t|y,y] \cdot (1-\rho)\pi_y \\
&+ \psi E[[1-(1+\nu)\tilde{c}_z](1+r_{t+1})\tilde{w}_t|y,o] \cdot q\pi_o + w_e q\pi_o \\
&= (1-\rho) \cdot [1-(1+\nu)\tilde{c}_y] \cdot (1+\bar{r}_{t+1}^{y|y}) \cdot E[\tilde{w}_{yt}]\pi_y \\
&+ \psi q \cdot [1-(1+\nu)\tilde{c}_o] \cdot (1+\bar{r}_{t+1}^{y|o}) \cdot E[\tilde{w}_{ot}]\pi_o + q w_e \pi_o \\
&= (1-\rho) \cdot [1-(1+\nu)\tilde{c}_y] \cdot (1+\bar{r}_{t+1}^{y|y}) \cdot \tilde{W}_{y,t} \\
&+ \psi q \cdot [1-(1+\nu)\tilde{c}_o] \cdot (1+\bar{r}_{t+1}^{y|o}) \cdot \tilde{W}_{o,t} + qW_e
\end{aligned}
\tag{32}
$$

Equations (31) and (32) form a system of laws of motion that determines the evolution of the wealth distribution

$$
\begin{pmatrix} \tilde{W}_{y,t+1} \\ \tilde{W}_{o,t+1} \end{pmatrix} = \begin{pmatrix} qW_e \\ 0 \end{pmatrix} + M \cdot \begin{pmatrix} \tilde{W}_{y,t} \\ \tilde{W}_{o,t} \end{pmatrix}
\tag{33}
$$

where the matrix $M$ collects transition probabilities, marginal propensities to save out of wealth, and expected portfolio returns correspondingly. Hence, the stationary levels of total wealth by household type are

$$
\begin{pmatrix} \tilde{W}_{y} \\ \tilde{W}_{o} \end{pmatrix} = (I-M)^{-1} \cdot \begin{pmatrix} qW_e \\ 0 \end{pmatrix}
\tag{34}
$$