Market Depth, Leverage, and Speculative Bubbles*

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Abstract

We develop a model of rational bubbles based on leverage and the assumption of an imprecisely known maximum market size. In a bubble, traders push the asset price above its fundamental value in a dynamic way, driven by rational expectations about future price developments. At a previously unknown date, the bubble will endogenously burst. Households' decision to lend to traders with limited liability in a bubble is endogenous. Bubbles reduce welfare for future generations. We provide general conditions for the possibility of bubbles depending on uncertainty about market size, traders' degree of leverage and the risk-free rate. This allows us to discuss several policy measures. Capital requirements and a correctly implemented Tobin tax can prevent bubbles. Implemented incorrectly, however, these measures may create the possibility of bubbles and can reduce welfare.

Keywords: Bubbles, Rational Expectations, Market Size, Liquidity, Financial Crises, Leveraged Investment, Capital Structure.

JEL-Codes: E44, G01, G12.

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1 Introduction

Which conditions can lead to bubbles? How can they be prevented? In the light of their repeated occurrence and the ongoing policy debate regarding potential counter-measures, these questions seem important and topical. We propose a fundamental mechanism that facilitates the emergence of bubbles, based on an unknown maximum market size and limited liability of traders. Analyzing optimal lending of households to traders and the investment decision of traders, we derive conditions under which bubbles can occur. These conditions allow us to evaluate policy measures for the prevention of bubbles, such as capital requirements and a Tobin tax. We also analyze who benefits from these policies and who loses.

The model features traders and households, both with endowments that they want to invest. Households have a relative investment disadvantage, which incentivizes them to lend to traders. We make two crucial assumptions in the model setup. First, traders face limited liability towards households. Second, the number of households and traders is fixed but unknown. We call the aggregate volume of traders' and households' endowments *market depth*, as it represents the maximum amount of money that a specific asset market can attract. Rational traders in this market are willing to invest in an overpriced asset, as long as there is a sufficiently high probability that they will be able to sell the asset, at an even higher price, to yet another future market participant.[1] In this case there can be a price path above the steady-state price, which we call a bubble. At some point, the bubble will exceed market depth, leading to an immediate burst. The risk of holding the asset at this moment deters traders from buying the asset on their own account. In equilibrium, however, traders are leveraged with loans from households. This raises incentives to invest in a risky, overpriced asset. Whether this rationale is sufficient to create a bubble depends on the thickness of the tail of the ex-ante distribution of market depth and on the degree of leverage.

By assuming an unknown market depth and limited liability, we combine core elements from Blanchard and Watson (1982) and Weil (1987) on the one hand, and Allen and Gale (2000) on the other. Blanchard and Watson show, in a nutshell, that asset-price bubbles can always emerge if market depth is implicitly assumed to be large enough.[2] Weil (1987) and related papers demonstrate that bubbles can also develop if the economy, and hence

[1] In the model, traders are aware that they are investing in a bubble. Conlon (2004) argues that in many bubble periods the overvaluation of assets was widely discussed. Referring to the dot-com bubble, Brunnermeier and Nagel (2004) provide evidence that hedge funds were riding the bubble, a result similar to a previous finding by Wermers (1999). The authors relate this to a short-term horizon of the managers, among other elements. Our model is consistent with this notion.

[2] Tirole (1982) takes the opposite position and shows that in a simple finite economy, bubbles are impossible. Also Santos and Woodford (1997) demonstrate that the conditions for the existence of bubbles are very restrictive if one assumes a fixed number of households that participate in the asset market and own finite aggregate endowments. The model of Zeira (1999) is similar in spirit to our model as he also assumes an unknown market size after, e. g., a financial liberalization. This uncertainty, however, creates asset-price booms and crashes by moving the fundamental value, above which the price cannot rise. Similarly, Allen and Gale (2000) show in a two-period model that expected expansions in credit can generate uncertainty about the steady-state price, which influences prices in previous periods. Prices can then also fall, depending on the realized expansion of credit.

market depth, is growing in a dynamically inefficient way. Due to the assumption of a *fixed* market depth, in contrast, our model applies to short-run dynamics. That is, bubbles can emerge in a stationary economy and their expected lifespan may be brief. Moreover, and different from Blanchard and Watson (1982), we explicitly spell out the conditions that beliefs about market depth have to fulfill for bubbles to be possible. Crucially, no matter how large the bubble has become, there must always be at least some probability that it will survive until the next period. Thus, the distribution function of ex-ante expectations regarding market depth needs to have unbounded support. Yet, market depth is finite and will be revealed once reached. Traders learn from each price increase that market depth has not yet been exhausted and update their expectations accordingly. For a low degree of uncertainty about market depth, the bubble size that will soak up all market liquidity can be computed with sufficient accuracy. When the asset price is close to this maximum, traders will be unwilling to continue investing. By backward induction, no bubble can then exist in the first place. For a higher degree of uncertainty, bubbles can emerge but will endogenously burst at an unknown date. Our assumption of finite but imprecisely known market depth therefore endogenously determines a time-varying probability of bursting; in Blanchard and Watson (1982) and Weil (1987), this probability is constant and exogenous. Unknown market depth in our model is meant to capture, in a stylized way, uncertainty regarding the size of future investments into a specific market. The financial crisis of 2007 has forcefully shown that domestic and, even more so, international capital flows can swell and dry up very quickly. The size and end point of these flows are not precisely predictable, just like the fraction that will be channeled into a specific asset.[3] Hence, in light of increasingly complex and opaque financial markets, the maximum amount of resources that a particular market can attract has to be estimated, with significant uncertainty always remaining.

Yet, even for relatively high uncertainty regarding market depth, traders in the model are only willing to invest in a bubble if they do not bear the full financial risk in case of a burst. Like Allen and Gale (2000), we hence find that traders are willing to invest in overpriced assets if some of the risk is shifted towards households. Different from Allen and Gale, these overvaluations may increase dynamically above the steady-state price, which is constant and known to all agents. In our model, limited liability arises because traders borrow from rational, risk-neutral households in addition to investing their own funds.[4] Households know whether the market is in a bubble but cannot observe the investment decisions of traders. Instead, we assume that households can monitor project returns only at a cost. This endogenously makes debt the preferred form of a financial contract, although households know that leveraged traders invest in assets they would deem too risky from their private perspective. Traders profit from rising asset prices, but potential losses are limited to their own invested funds. The model therefore directly

[3] Before the crisis, many foreign investors were buying collateralized debt obligations issued in the US. How much they had invested was not even clear for market participants after the crisis had unfolded (see, e. g., Carrington, Coleman, Sloman, and Blumenthal, L.L.P. 2008). More recently, many observers are wondering how much Chinese investors in particular will continue to invest in the Australian housing market, see UBS (2017) and Punwasi (2017), among others.

[4] According to the OECD database on institutional investors' assets, in 2007 institutional investors in the US managed assets worth 211.2% of GDP, showing investors' prominent role in investment decisions. Furthermore, the assets' size has grown steadily over the last decade with a yearly average growth rate of 6.6% from 1995-2005 within the OECD(17) area (see Gonnard, Kim, and Ynesta, 2008).

applies to any type of intermediated finance with limited liability, such as investment through banks, investment banks, insurance companies, and private equity firms as well as to non-intermediated, debt-financed investments. We show that, depending on traders' investment opportunities compared with those of households, the latter are willing to lend to traders under similar conditions as those that let traders invest in bubbles. We also demonstrate that traders who start a bubble benefit in terms of realized consumption, while households that invest later have lower expected consumption levels if a bubble has emerged. This effect arises because limited liability of traders leads to a socially suboptimal degree of risk taking; it increases as the bubble grows. Policy interventions to prevent bubbles, if successful, can hence increase expected welfare of future investors.

Summing up, the first main contribution of our paper is the development of our bubble-generating mechanism by using elements from Blanchard and Watson (1982), Weil (1987), and Allen and Gale (2000). Neither heterogeneous traders, dynamic inefficiency, expected or realized changes in fundamentals, nor a stochastic or positive growth rate of the economy are necessary for bubble creation.[5] In this setup, the possible existence of bubbles depends on the economic environment because, first, bubbles might become too risky for traders to invest and, second, households' participation constraints can be violated, today or at a future date. We provide necessary and sufficient conditions for both constraints. Hence, depending on the interaction of leverage, uncertainty about market depth, riskiness of the asset, and the risk-free interest rate, the prerequisites for bubbles may be fulfilled or not.[6] The model can hence answer questions like 'Does high leverage foster the emergence of bubbles?' Some policy implications ensue immediately; they form the second main contribution of the paper.

The remainder of this paper is organized as follows. Section 2 introduces the model and then develops a steady-state (rational-expectations) equilibrium price process. Section 3 provides a necessary and sufficient condition for the existence of bubbles. The section begins with the construction of a special type of bubble, which then serves as an example for the general case. Section 4 provides a necessary and sufficient condition for households' participation. The conditions lend themselves to basic policy analysis, which is performed in Section 5, beginning with a welfare analysis. Section 6 concludes. Appendix A discusses welfare with heterogeneous households. All proofs are in Appendix B.

[5] Brunnermeier (2001) provides an extensive survey of bubble models based on asymmetric information. From the vast literature on bubbles, Allen and Gorton (1993) is closest to our model in spirit. In their finite-horizon model in continuous time, traders are exiting the bubbly market one after another. They are willing to take on the risk of being the last market participant because of limited liability. Heterogeneous types of traders and stocks induce investors, who cannot buy the same assets, to lend despite limited liability. For each trader, the number of remaining other traders is uncertain. Different to our model, higher risk does not imply a higher return for traders, as they do not invest own funds. The resulting dynamics are quite different to ours.

[6] Using the latest US housing bubble as an example, we find that conditions that are favorable for the emergence of bubbles in our model were fulfilled. Increasingly international financial flows and more complex financial instruments obscured potential market depth. Furthermore, the Securities and Exchange Commission's 2004 decision to allow large investment banks to assume more debt raised their leverage and further increased uncertainty about market depth. Kaminsky and Reinhart (1999), among others, also indicate an empirical connection between financial liberalization, credit expansion, and bubble emergence. Moreover, Adelino, Schoar, and Severino (2015) find evidence that house buyers were indeed attracted by the prospect of higher future prices.

2 The Model

2.1 Setup

Consider an infinite-horizon economy with a series of cohorts of risk-neutral households and traders.[7] In each period, a continuum of households and a continuum of traders are born, both of measure N. Each household has an initial endowment of $l \in (0,1)$, and each trader owns $e = 1 - l$ dollars, such that the total endowment of one household and one trader is normalized to \$1. N is therefore the amount of wealth in the economy, and thus the maximum price of any asset. We refer to this concept as market depth. It is fixed over time but unknown. The probability that this aggregate endowment takes a specific value N has the distribution $F(N)$ with unbounded support. The relevant statistic of the distribution function for the mechanics of the model is a relative version of the hazard rate $N\,f(N)/(1 - F(N))$, which denotes the probability that market depth will be exhausted for a marginal percentage increase of the price. This probability is important for traders who decide whether to invest in the risky asset. By employing a function $F(N)$ with unbounded support, we implicitly assume that there is always the possibility of a further marginal price increase, given that the price has already evolved to some current level. This prediction seems realistic: traders can never be absolutely sure that not even a single additional cent will flow into a given asset.[8] This setup can be seen as a shortcut to a situation in which traders are uncertain about how many resources other traders are still willing to invest in a certain market. We make the technical assumption that the density $f(N)$ exists and that $F(N)$ follows a Pareto distribution,[9]

$$F(N) = 1 - (N/N_0)^{-\gamma} \tag{1}$$

for $N \in [N_0, \infty)$, with $N_0 > 0$ and $\gamma > 0$. Here, γ is the above discussed relative hazard rate, it measures the thinness of the tail or equivalently the precision of the information on market depth N. If $\gamma \to \infty$, $N = N_0$ is common knowledge. We will see that γ plays a crucial role in the evolution of bubbles, whereas N_0 drops out of the analysis at an early stage.

[7] Traders enter and exit the market in an OLG fashion to generate trade each period. We do not see this OLG structure as representing actual generations, but as a shortcut for non-modeled market imperfections, such as heterogeneous liquidity preferences of traders.

[8] In a model with stochastic growth, we would similarly obtain a non-zero conditional probability for future increases of total resources. In this case, N would vary stochastically over time. The present setup with a fixed N, in contrast, corresponds to an analysis of short-run dynamics. Note that infinite horizons are not central to this model setup in which traders hope to sell a risky asset before the bubble bursts, if combined with an asymmetric information framework similar to Allen, Morris, and Postlewaite (1993), Conlon (2004, 2015), and Doblas-Madrid (2016).

[9] Our results also hold if the distribution is not Pareto. In this case, the parameter γ has to be replaced by $\lim_{N\to\infty} N\,f(N)/(1 - F(N))$, if this limit exists. To be precise, if the convergence of the limit is monotone from above, all results hold. If it is not, bubbles still exist if the below inequality (12) is strict. The additional insight regarding the intuition of the model seems marginal, we abstain from a formal discussion.

Households and traders have a life span of two dates. They invest at date t. At date $t+1$, they earn their return, disinvest, and consume. The duration of a period between two dates stands for the investment horizon of a trader. Traders and households have identical utility functions: $U = C_t + \rho\, E_t C_{t+1}$. There are two types of assets, safe assets (short: storage) of unlimited supply and a single risky asset (short: the asset) with a supply of one. Storage bears a risk-free return of $Y > 1$ to a trader. The return of the storage asset to a household is only λY, with $0 \leq \lambda < 1$ and $\rho \lambda Y > 1$, such that the payoff from investing dominates immediate consumption of the endowment. The inverse of λ thus measures the investment advantage of traders, which may arise because of better abilities (e. g., due to optimal selection into professions) or better information. Only traders have access to the risky asset.[10] The asset can be interpreted as shares of a firm. It cannot be shorted. The firm pays a dividend of d in each period. There is a probability $1-q \geq 0$ in each period that the firm goes bankrupt and ceases to pay dividends forever; the firm's shares are then no longer traded.[11] The risky asset is traded in a competitive market in each period. The price of the asset $\{\tilde{p}_t\}_{t \geq 0}$ is endogenous and may be stochastic.

2.2 The contract between households and traders

Households can lend to traders in order to profit from their better investment opportunities. Both parties have to agree upon a repayment structure. We assume that households propose the contract, hence they have the bargaining power. In general, if the investment's gross return is R, then the financial contract between the two can stipulate a repayment of any $z(R) \leq R$. For example, $z(R) = \alpha\, R$ (with $0 \leq \alpha \leq 1$) can be interpreted as a household buying shares of a fund that is run by a trader. $z(R) = \min\{\beta; R\}$ (with $\beta \geq 0$) can be interpreted as a standard debt contract between trader and household. Because each household is small, it does not internalize the externality of its contract choice on equilibrium prices. A household cannot observe the return from the traders' investment. It can, however, engage in costly monitoring to verify the true return. This entails a cost of $c > 0$. Commitment to monitoring, contingent on the trader's repayment, is possible.[12]

[10] Even if households could invest in the risky asset, they would not do so because in equilibrium the asset will be overpriced (as we will show in Section 2.4). Note that the reason for trade between agents is the higher return of the traders, as in Allen and Gale (2000), Allen and Gorton (1993), and Barlevy (2014), instead of some form of risk sharing, as in Allen, Morris, and Postlewaite (1993) and Conlon (2004, 2015).

[11] One may also interpret the asset as real estate. If a house is used as rental property, d denotes the rent per period, whereas $1-q$ is the probability that the house becomes uninhabitable. The parameter q will alter the condition for the emergence of bubbles. However, $q = 1$ (no risk) is also admissible for the following derivations.

[12] Like Townsend (1979) and Gale and Hellwig (1985), we consider commitment only in pure strategies. Implicitly, we thus assume that a commitment device exists, but only for pure strategies. If stochastic monitoring were allowed, households could save money by an expected monitoring frequency just high enough to implement truth telling on the side of the trader. The optimal contract may then involve rebates, like in Border and Sobel (1987). It can, however, also resemble a standard debt contract, like in Cole (2013), depending on the exact set of assumptions. Importantly, the trader's expected return is convex in her investment success. Hence, in neither of these modeling choices would our key ingredient for bubbles disappear, but the algebra would be much more involved.

In analogy to Gale and Hellwig (1985), we can then show that the optimal contract under pure strategies has the shape of a pure debt contract (see also Townsend 1979). The rest of the model is independent from the specific endogenization; for the emergence of bubbles, any other micro-foundation of debt contracts would be equivalent.

Lemma 1 *There is a* $\bar{c} > 0$ *such that for* $c \geq \bar{c}$ *the household chooses a contract of the form*

$$z(R) = \min\{\beta; R\}. \tag{2}$$

The trader then gets the residual

$$\max\{R - \beta; 0\}. \tag{3}$$

The parameter β represents the contracted repayment of a loan, including interest payments. If the loan size is l (equal to $1-e$) and the loan rate is r, β results as $\beta = r\,l$. It can also be interpreted as a hurdle: if the return exceeds β, the trader collects a bonus, otherwise not.[13] This parameter will be set endogenously in the negotiations between households and traders. The variable β subsumes the contract. To analyze the consequences of the contract design, we will discuss how bubbles depend on β, but bear in mind that β depends both on leverage l and on the opportunity interest rate Y.

2.3 Equilibrium

We solve for stochastic rational-expectations equilibria. A stochastic process $\{\tilde{p}_t\}_{t\geq 0}$ must be such that, given that households and traders have rational expectations and behave individually rationally, the market for the asset clears. The equilibrium need not be unique. At each date, old traders sell the asset to new traders. The market works in a typical Walrasian fashion. New traders submit their individual demand function to a market maker. Old traders submit their supply: because they have to sell in order to consume, they will sell at any price. An auctioneer then sets the price that clears the market and allocates the shares of the asset accordingly.

Before analyzing the model, let us summarize the key frictions and their implications. First, households have an investment disadvantage relative to traders, i. e., they receive a lower return when investing in the safe asset. They hence try to benefit from the better investment opportunity of traders. Second, ex interim, households cannot monitor traders' investment choices. Third, households cannot monitor investment returns without costs (costly state verification). These three assumptions result in an endogenous debt contract. Fourth, market depth N is unknown. Each of these frictions is necessary for bubbles to emerge in our setting. Without the first friction, households would rather invest themselves

[13] With such an interpretation, other policy measures can be analyzed, such as the relation between the traders' compensation package and the emergence of bubbles.

directly. With the first friction but without the costly state verification, households would lend to traders using an equity contract. For traders, the risk of a bursting bubble would then outweigh the potential gains from a higher resale value. Third, if households could observe investments of traders without costs, they would simply prevent investments in bubbly assets, such that no bubbles could emerge. Finally, and most importantly, if the true value of N was known, the maximum market size could be calculated and backward induction would prevent a bubble from taking off. This highlights the crucial interaction between limited liability and uncertainty about market depth.

2.4 The Steady-State Price – Asset Market Clearing

In the following, we first analyze the steady state and then potential bubble paths. We start with discussing the traders' behavior and then check the households' participation constraint.

Consider the following simple stochastic process $\{\tilde{p}_t\}_{t\geq 0}$. The price of the asset is a constant, $\tilde{p}_t = \overline{p}$. It drops to zero only if the underlying firm goes bankrupt (with probability $1-q$), and cash ceases to flow. Hence, the price follows a simple binomial process with $\Pr_t\{\tilde{p}_{t+1} = \overline{p} | p_t = \overline{p}\} = q$. Zero is an absorbing state. The probability that an investment fails is thus independent of β. Since we assume that households have the bargaining power, β will be set such that traders are at their participation constraint: investing their own funds (expected return $Y\,e$) or investing their own plus borrowed money (expected return $Y - \beta$) yields the same expected payoff.[14] Hence,

$$\beta = r\,l = Y\,l = Y\,(1-e). \tag{4}$$

This argument carries over to markets with bubbles, as we will explain in the according proofs. Let us derive the price $\overline{p}$ for which this process is a rational-expectations equilibrium. In a market equilibrium, prices must be such that traders' expected return is the same for storage and for the risky asset. If a trader opts for storage, her payoff is $\max\{Y-\beta; 0\} = Y - \beta$ because $\beta = Yl \leq Y$. If the trader invests e own and $l = 1-e$ borrowed dollars into shares of the firm at a price $p_t = \overline{p}$, she can buy up to $1/\bar{p}$ shares. She benefits from the dividend d with probability q and hence earns d/p_t with the same probability. In the absence of a bankruptcy, the price remains at $p_{t+1} = \overline{p}$ and the trader additionally receives $p_{t+1}/p_t = \overline{p}/\overline{p} = 1$ from selling the asset. In the steady state, a trader's expected payoff from holding the risky asset at date t is therefore

$$E_t \max\left\{0;\, \left(\frac{\tilde{p}_{t+1}}{p_t} + \frac{d}{p_t} - \beta\right)\right\} = q\left(\frac{\overline{p} + d}{\overline{p}} - \beta\right)$$

[14] The expected return for an unleveraged trader is $Y\,e$, independent of the form of investment. If other traders are leveraged, they push the asset price to a level that makes an investment unattractive for unleveraged traders, who prefer storage instead. In the case that all traders are unleveraged, the asset price adjusts such that they are indifferent between both. In this context, we can therefore use the payoff Y from storage without loss of generality.

For the market to clear, traders must be indifferent between storage and the asset,

$$Y - \beta = q\left(\frac{\overline{p} + d}{\overline{p}} - \beta\right) \tag{5}$$

$$\Rightarrow \quad \overline{p} = \frac{d\,q}{Y - q - \beta\,(1-q)}. \tag{6}$$

The steady-state price $\overline{p}$ thus depends on traders' leverage l (as $\beta{=}Y\,l$). The fundamental value represents the expected discounted future dividends

$$\underline{p} := \sum_{t=1}^{\infty} \frac{q^t\,d}{Y^t} = \frac{d\,q}{Y - q} \leq \overline{p}. \tag{7}$$

Hence, only if $\beta{=}0$ (no leverage) or if $q{=}1$ (no fundamental risk), the fundamental value and the steady-state price are equal, $\underline{p} = \overline{p}$. A $\beta > 0$ makes the traders' target function convex, which raises the asset price in the presence of risk. The deviation between the two is static and driven by fundamentals, but not by traders' expectations about future price developments. The effect of leveraged traders pushing prices of risky assets above their fundamental levels has been analyzed previously by Allen and Gale (2000). Similarly, Malamud and Petrov (2014) have shown that convex incentives lead to mispricing in the form of prices above the fundamental value of an asset.

Traders' Bids. On the basis of equation (5) we can derive the traders' demand function for the asset in more detail and analyze how the market clears. Let us assume that traders, before bidding for the asset in period t, expect the steady-state price $\bar{p}$ to be realized. Each trader has 1\$. At the price $\bar{p}$, they can hence buy at most a volume $1/\bar{p}$ of the asset. For a volume of shares $v > 1/\bar{p}$, each trader could not pay more than one dollar, such that the price would be $1/v < \bar{p}$. This is cash-in-the-market pricing: the demand function is downward sloping. Now consider a volume $v < 1/\bar{p}$. After paying $p\,v$ for the asset at the price p, the trader can additionally store $1{-}p\,v$, yielding a safe rate of return of $Y\,(1{-}p\,v)$. If the firm defaulted, she would have to use this cash to pay out households. Taking this into account, her expected return would have to equal that from buying the volume $1/\bar{p}$ at the price $\bar{p}$ and selling at $\bar{p}$,

$$q\Big(v\,(\bar{p}+d) + (1-p\,v)\,Y - \beta\Big) = q\left(\frac{\bar{p}+d}{\bar{p}} - \beta\right),$$

$$p = \frac{1}{v} - (1-\bar{p}\,v)\,\frac{\bar{p}+d}{\bar{p}\,v\,Y}. \tag{8}$$

For $v = 1/\bar{p}$, this yields $p = 1/v = \bar{p}$, and for $v < 1/\bar{p}$, we have $p < 1/v = \bar{p}$. In this region, the demand function of a trader is backward sloping: for a lower volume, she will offer a *lower* price per share. The rationale for this effect is the traders' de facto preference for risk, caused by her limited liability and the fact that her own investment in storage is lost in case of a bankruptcy of the firm. Traders hence do not like to mix both forms of investment. As a consequence, a trader's demand function $p(v)$ has a maximum at $v = 1/\bar{p}$. This implies that, in equilibrium, a volume of $\bar{p}$ traders will get $1/\bar{p}$ shares allocated, while the others will store their dollar. A more dispersed allocation would lead to lower prices and would hence be overbid by traders that ask for a larger share. As another consequence, traders cannot infer their number from the market.

2.5 The Steady-State Price – Household Participation

We now check whether households want to lend to traders at the steady-state price. A household investing $l = 1 – e$ can always obtain $\lambda\,(1 – e)\,Y$ from storage. When the household delegates investment to a trader, it does not know whether the trader uses the money to buy the risky asset. At an asset price $\overline{p}$, a number of $\overline{p}$ traders buy the asset, while the remaining $N – \overline{p}$ invest safely (storage). The probability that a trader invests riskily is thus $\overline{p}/N$. In this case, the household is repaid β only with probability q. With probability $1 – q$, the trader defaults and the household pays the verification cost c. If the trader stores the capital and earns a gross return Y, the household receives β with certainty without any verification cost. Households, however, do not know N. They know that N must be at least $\overline{p}$, given that the price $\overline{p}$ has already been paid in the past. The conditional distribution is $F(N|p = \overline{p}) = 1 - (N/\overline{p})^{-\gamma}$, and thus $f(N|p = \overline{p}) = \gamma\, N^{-\gamma-1}\,\overline{p}^{\gamma}$. The expected return to a household in steady state is then

$$\int_{\overline{p}}^{\infty} \left[\frac{\overline{p}}{N}\,(q\,\beta - (1-q)\,c) + \left(1 - \frac{\overline{p}}{N}\right)\beta \right] f(N)\,\mathrm{d}N = \frac{(1+q\,\gamma)\,\beta - (1-q)\,\gamma\,c}{1+\gamma}. \tag{9}$$

This expected return exceeds the household's private return $\lambda\, Y\,(1 - e)$ whenever

$$\lambda \leq \bar{\lambda} := \frac{1}{l\,Y}\,\frac{(1+q\,\gamma)\,\beta - (1-q)\,\gamma\,c}{1+\gamma}. \tag{10}$$

If $\lambda > \bar{\lambda}$, gains from trade are so small and the risk of speculative investment is so large that households refrain from lending to traders – the loan market breaks down. As a consequence, traders are not leveraged and the asset trades at its fundamental value.

3 Bubbles – Asset Market Clearing

We now come to the main question of the paper. For given fundamentals, is an alternative price path possible, with a price above the steady-state price $\overline{p}$? If it is, we call this path a bubble.

Definition 1 (Bubble) *In our model, a bubble is defined as a price process in a rational-expectations equilibrium in which the price deviates from the steady-state price, i. e., from the leverage-adjusted present discounted value of future dividends.*

The steady-state price process is always an equilibrium, hence if a bubble exists, there are multiple equilibria by definition. Our definition captures unsustainable price developments that are based on expectations about future price increases. The only reason for a trader to buy the risky asset at a price above the steady-state price is that she expects the price to rise even further, at least with some probability. As the deviation between the steady-state price 6 and the fundamental value 7 is static and not driven by expectations about rising prices, the steady-state price is the relevant benchmark for our definition of bubbles. The following theorem answers our main question.

Theorem 1 *In a rational-expectations equilibrium, the asset market* can *exhibit a bubble if and only if*

$$(\gamma+1)(1-\beta)<1 \qquad \textit{and} \tag{11}$$

$$Y-\beta \leq q\,\frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}. \tag{12}$$

The proof of Theorem 1 proceeds in two steps. In Section 3.1, we first assume that (11) and (12) hold and construct a bubble. We then show that, if the conditions do not hold, the steady state is the unique price path, such that bubbles are not possible. The intuition for the proof is in the main text, some of the formalism is relegated to Appendix B. The following proposition summarizes the comparative statics.

Proposition 1 *If the bubble conditions* (11) *and* (12) *hold for given values of* Y, q, γ, l, λ, *and* c, *then they also hold for any lower values of* Y, γ, λ, *and* c. *They also hold for any larger values of* q *and* l.

In this sense, bubbles tend to be possible for a low risk-free yield Y, low fundamental risk (large q), large uncertainty about market depth (low γ, see Figure 1), and high leverage (high l, see Figure 1, with $\beta = Yl$).[15] In fact, for $\gamma \to 0$, we get the result of Blanchard (1979) and Blanchard and Watson (1982): the asset market can always exhibit bubbles. For $\gamma \to \infty$ (certain market depth), we obtain Tirole (1982)'s impossibility result, see footnote 2. Note the difference between a *dynamic* price deviation from the steady-state price and the *static* deviation of the steady-state price from the fundamental value. The static deviation is larger for inherently risky assets, but bubbles tend to emerge for inherently safe assets. In this context there is a subtle but important difference between the inherent and the financial risks of an asset. To provide an example, a house may be an inherently safer investment than stocks of a firm, at least outside earthquake regions. Considering financial risk, however, a house may be a riskier investment if it is built during a bubble. Our model distinguishes between these notions of risk: inherent risk is captured by $1-q$, the risk of failure of the underlying asset.[16] Additional financial risk occurs if conditions (11) and (12) hold and a bubble can emerge and burst.

[15] Note that Y has a double role: A higher Y directly reduces the range for bubbles, but it increases β, which indirectly extends the parameter range with bubbles. It is straightforward to show that the direct effect dominates (see proof of Proposition 1).

[16] Note that a change in q automatically influences the asset's expected payout. In order to vary the

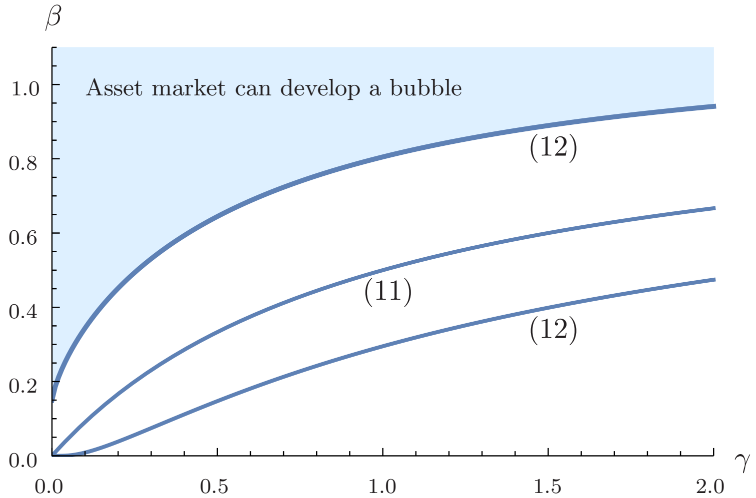

Figure 1: Parameter Range where Bubbles are Possible on the Asset Market

The figure is based on a numerical example with $Y = 1.1$ and $q = 0.95$. Condition (11) holds above the middle curve; below that curve, the information on market depth is too precise (high γ), i. e., the probability of a burst of a bubble is too large and traders are not willing to buy an overpriced asset. Condition (12) holds above the upper curve and below the lower curve (solving for β yields two solutions). in between, the overpriced asset is dominated by storage at some point in time, such that no bubble can emerge. Combining conditions (11) and (12) shows that – from the traders' perspective – bubbles are possible in the shaded parameter region.

3.1 Construction of a Bubble

We now construct an example bubble, that way proving that a bubble can exist if (11) and (12) hold. Consider a special ("trinomial") class of price processes with

$$\tilde{p}_{t+1} = \begin{cases} 0, & \text{with probability } 1 - q \\ \overline{p}, & \text{with probability } q - \bar{Q}_t \\ p_{t+1}, & \text{with probability } \bar{Q}_t \end{cases} \tag{13}$$

with $\bar{Q}_t \leq q$.[17] The sequence of variables $\{p_t, \bar{Q}_t\}_{t \geq 0}$ will be determined endogenously. Trinomial processes are the simplest ones that allow for a fundamental default of the firm (first case in equation 13), a bursting of the bubble (second case), and the continuation of the bubble (third case). A possible price process is depicted in Figure 2. In the figure, the process begins at some price $p_0 > \overline{p}$ above the steady-state price. At each date, the price can develop in only three ways: the three cases of (13).

asset risk without changing the expected payout, one might want to adjust d when changing q. One could keep either $q\,d$, the expected payoff $d\,q/(1-q)$, or the fundamental value constant. Because the dividend d does not enter the conditions of Theorem 1, however, the comparative statics of Proposition 1 would not change at all.

[17]Note the notational difference between $\tilde{p}_{t+1}$ and p_{t+1}. $\tilde{p}_{t+1}$ is the stochastic price at date $t+1$ that can assume three different values. $p_{t+1} > p_t$ is the largest of these realizations.

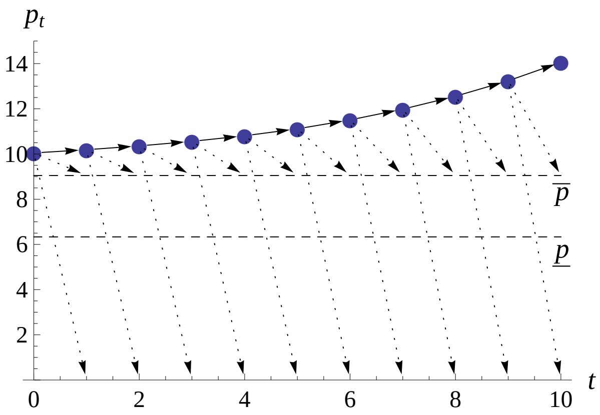

Figure 2: A Trinomial Price Process with a Bubble

Here and in the following figures, the parameters are $\beta = 0.9$, $q = 0.95$, $d = 1$, $Y = 1.1$, and $\gamma = 1$.

If a bubble follows a trinomial price path, as long as the bubble doesn't burst, the price grows further and further, $p_0 < p_1 < p_2 < \ldots$ At any price p_t, because each trader disposes of one dollar, a number of p_t traders invests in the bubble. This implies that more and more money will be absorbed by the bubble. At some date, p_t will reach N. As N is unknown, also this date is not known. At this point, the investments of all traders are insufficient to absorb the entire asset for prices above N, i. e., the market cannot clear. Traders realize by backward induction that no price above $\overline{p}$ can be sustained. They hence stop demanding the asset and no future generation offers $p_t > \overline{p}$. In short, from the moment on that an upper ceiling for N is revealed, the only possible price is the steady-state price. No more bubbles can occur until developments generate new uncertainty about market depth. The date at which the bubble bursts is (and must be) unknown, but the ceiling will be reached with certainty at some date.

The price that a trader is willing to pay depends on the expected future price increase and on the probability of a burst, which can be calculated from the distribution of N, as perceived by the trader. This subjective distribution will be conditional on two facts. First, for a current price p_t at date t, the trader knows that $N \geq p_t$. Second, if the number of traders is large, the probability that a trader is allocated shares in the market is small. Hence, the trader gets a piece of information on the size of N at the time she is allocated a share (or not). Let X mark the event that a trader bids successfully for the asset, given symmetric bidding strategies. Then

$$f(N|X) = \frac{\Pr(X|N)\, f(N)}{\int_{p_t}^{\infty} \Pr(X|N)\, f(N)\, dN} = \frac{\frac{p_t}{N}\,\gamma\,\frac{N^{-\gamma-1}}{p_t^{-\gamma}}}{\int_{p_t}^{\infty} \frac{p_t}{N}\gamma\,\frac{N^{-\gamma-1}}{p_t^{-\gamma}} dN} = (\gamma+1)\frac{N^{-\gamma-2}}{p_t^{-\gamma-1}}. \tag{14}$$

The conditional probability that N is between p_t and p_{t+1} (which means that the bubble will burst in the next period) is then

$$\Pr\{N \leq p_{t+1} | N \geq p_t, X\} = \frac{F(p_{t+1}|X) - F(p_t|X)}{1 - F(p_t|X)} = 1 - \left(\frac{p_t}{p_{t+1}}\right)^{\gamma+1}, \tag{15}$$

as $p_{t+1} > p_t$. If p_{t+1} was below p_t, the probability of reaching the ceiling would be zero because $N \geq N_0 = p_t$ for sure. Remember that the bubble can also burst because the firm goes bankrupt (probability $1-q$). In this case, no dividends are paid and the shares are no longer traded; the profit of the trader is 0. Hence, the probability that the bubble does *not* burst (neither because of the ceiling nor because of bankruptcy), given that the trader will bid successfully, is

$$Q_t = q\left(1 - \frac{F(p_{t+1}|X) - F(p_t|X)}{1 - F(p_t|X)}\right) = q\,\frac{p_t^{\gamma+1}}{p_{t+1}^{\gamma+1}}. \tag{16}$$

For $p_{t+1} \leq p_t$ it would be q. The asset market can only be in equilibrium if a modified version of the arbitrage condition (5) holds: a trader's profit from storage $Y-\beta$ must equal the probability Q_t (bubble doesn't burst) times the expected profit, including appreciation and dividend yields,

$$Y - \beta = q\,\min\left(1; \frac{p_t^{\gamma+1}}{p_{t+1}^{\gamma+1}}\right)\,\max\left(0; \frac{p_{t+1}+d}{p_t} - \beta\right). \tag{17}$$

If the share price falls because market depth is exhausted, the price drops to $\overline{p}$. If this drop is large enough, the revenue generated by the trader is too small to fulfill her contractual commitment and at the same time keep profits for herself (the general case is discussed in Appendix B). That is, she needs to hand over all revenue to the household to provide at least parts of the agreed payment. By assuming that p_{t+1} is large, we can hence drop the min and max operators. The above equation gives an implicit recursive rule for the evolution of the price process. Starting with some $p_0 > \overline{p}$, the equation implicitly defines p_1, which is just high enough that the appreciation compensates a trader for buying an overpriced asset, i. e., for the risk of a bursting bubble.[18] We can then use (17) to calculate p_2 from p_1, and so on. One such process is shown in Figure 2. Starting from p_0, the complete process $\{\tilde{p}_t\}_{t\geq 0}$ is implicitly defined – at least if equation (17) has a solution.

Traders' Bids. We have argued that in the steady state the demand function of traders reflects the fact that they behave as if they were risk-loving, implying that they either want to buy only shares or no shares at all. This is also the case in a bubble. Corresponding to equation (8), traders are now indifferent between buying a volume $1/v$ at price p (per share) and the volume $1/\bar{p}$ at the price $\bar{p}$ if

$$q\,p_t^{\gamma+1}/p_{t+1}^{\gamma+1}\Big(v\,(p_{t+1}+d) + (1-p\,v)\,Y - \beta\Big) = q\,p_t^{\gamma+1}/p_{t+1}^{\gamma+1}\left(\frac{p_{t+1}+d}{p_t} - \beta\right),$$

$$p = \frac{1}{v} - (1 - p_t\,v)\,\frac{p_{t+1}+d}{p_t\,v\,Y}, \tag{18}$$

which takes the possibility of a bursting bubble into account. The rest of the argument is the same as before. In particular, traders cannot infer their number from the market price.

[18] Note that, starting with the steady-state value $p_0 = \overline{p}$, the path $p_t = \overline{p}$ for all t is a solution of the implicit equation – we remain in the steady state.

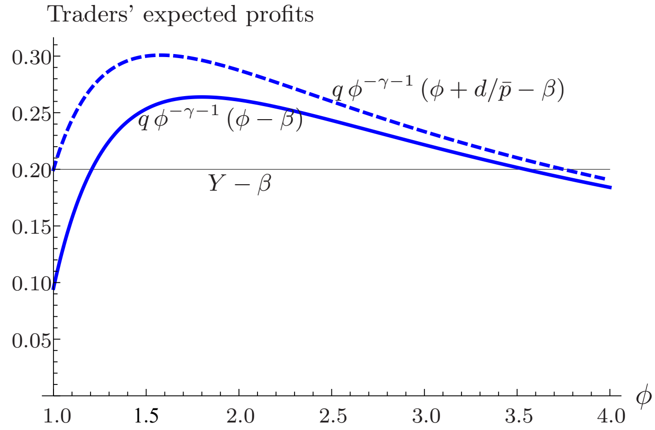

Figure 3: Possibility of a Bubble – Expected Returns

As in the previous figures, parameters are $\gamma = 1$, $\beta = 0.9$, $d = 1$, $q = 95\%$, and $Y = 1.1$.

Existence of the Bubble Process. Equation (17) does not necessarily yield a solution p_{t+1} for all starting points p_t. The higher the potential future price increase, the more likely it is that the ceiling N is reached and that the bubble bursts. The more likely the bubble is to burst, however, the larger the expected price increase must be to compensate traders for the risk they face. Hence, there might be no equilibrium price p_{t+1} for all initial prices p_t. If so, there exists a critical price above which the risk of a bursting bubble outweighs the potential gains from a price increase. Because all market participants can calculate the date t at which this critical price is reached (if it exists), a bubble would burst with certainty at t. By backward induction, the bubble is not sustainable right from the beginning – there is no bubble, the price path is unique. We are thus interested in conditions under which a bubble can or cannot be sustained. To be sustainable, the implicit equation (17) must have a solution at any date t or, equivalently, for any price p_t. Rewriting (17) by defining the auxiliary variable $\phi_t = p_{t+1}/p_t$ as the relative price increase yields

$$Y - \beta = q\,\phi_t^{-\gamma-1}\left(\phi_t + \frac{d}{p_t} - \beta\right). \tag{19}$$

The left-hand side of the equation is independent of p_t, but the right-hand side is not. Figure 3 shows the traders' expected profit from storage (left-hand side of equation 19, thin solid line) and the expected profit from buying the asset (right-hand side of 19, thick curves), which depends on ϕ_t. The expected profit from the asset depends also on the price at which the asset is bought. The dashed curve stands for the lowest possible price, the steady-state price $p_t = \overline{p}$. The solid curve represents the highest possible price, $p_t \to \infty$. Any intermediate price leads to a curve in between the solid and the dashed curve.

Figure 3 also contains a lot of economic intuition. *First*, the dashed curve intersects with the line at $\phi_t = 1$. At the steady-state price, traders do not need any asset appreciation as a compensation for buying the asset. *Second*, when the asset price increases, the curve moves downward. An intersection between the line and the curve can cease to exist, depending on the parameters. Hence, at some date t, the risk of a bursting bubble might be so large that traders can no longer be compensated by an asset appreciation (because that would further raise the risk of a burst). The bubble will burst with certainty and, by backward induction, no bubble can exist in the first place. To show that a bubble can be sustained in a market, it is thus sufficient to consider large prices p_t. In the limit of $p_t \to \infty$, (19) simplifies to[19]

$$Y - \beta = q\,\phi^{-\gamma-1}\,(\phi - \beta). \tag{20}$$

The equation does not depend on time; we have hence dropped the index t. *Third*, the curves are hump-shaped. This means that there can be several price increases ϕ_t that lead to market clearing. If ϕ_t is low, the probability of a burst is also low, and if the risk of a burst is high, the expected capital gain is high, too. Traders are thus indifferent at more than one value of ϕ_t. We are, however, mainly interested in the existence of a $\phi_t > 1$, so it does not matter how many solutions for ϕ_t we get. *Fourth*, is the curve always hump-shaped? Not necessarily. The slope of the curve at $\phi_t = 1$ is $q\,(\beta - \gamma + \beta\,\gamma)$, which is positive for $(\gamma+1)\,(1-\beta) < 1$, the first condition (11) of Theorem 1. In combination with the fact that curve always starts below the line, curve and line cannot intersect in the region of $\phi > 1$ if this condition is invalidated, i. e., if the slope is negative at $\phi = 1$. At the same time, there cannot be an equilibrium for $\phi < 1$, even if condition (11) is violated. Instead of the expected asset return given on the right-hand side of equation (20), which was derived for $p_{t+1} \to \infty$, equation (17) applies in this case. Because an expected price decrease reduces the payoff for traders while the probability of reaching the ceiling remains at zero for falling prices (which is contained in the min operator), the risky asset just becomes less attractive. To the left of $\phi = 1$, the curve has therefore a positive slope and no equilibrium is reached; traders prefer storage to the asset. Equation (20) has only an implicit solution, which results in condition (12).[20]

Given the above insights, let us discuss the intuition of (12), i. e., the comparative statics of Proposition 1. If the risk-free rate Y is higher, storage becomes more attractive to traders. Thus, they only hold the risky asset if it displays a larger potential price increase. A larger increase, however, corresponds to a higher likelihood of a burst, which might impede the existence of a fixed point. Hence, for a larger risk-free yield Y, bubbles might cease to be possible. This finding is consistent with the idea that central banks can puncture bubbles by raising interest rates and that bubbles are particularly likely if interest rates are low.[21] Furthermore, bubbles can exist particularly if q is high, that is, if the underlying asset

[19]The factor $\phi_t = p_{t+1}/p_t$ does not converge to infinity; it converges to a limit ϕ that is implicitly defined by (20). As a consequence, also the continuation probability of a bubble does not converge to zero but to $q\,\phi^{-\gamma-1}$.

[20]Note that there are two separate parameter regions that satisfy (20) (see Figure 1). Only the upper region, however, fulfills condition (11) as well. In the lower region, equation (20) has a solution, but with a $\phi < 1$. Formally, this is ruled out by (11).

[21]Note that this and related results stem from the fact that a higher growth rate of the bubble, induced by a higher alternative investment return, increases the risk of a burst. This relationship is itself an outcome of the assumption of a finite but unknown market size.

is rather safe, which reduces the likelihood of a burst due to a bankruptcy of the issuing firm. The parameter γ captures the uncertainty in the market. The smaller the value of γ, the larger are the mean and the variance of the distribution, and the more uncertain is the potential market size. For $\gamma \to 0$, the expected market depth becomes infinite. On the other hand, if $\gamma \to \infty$, the market depth is almost surely N_0 and a bubble can never be sustained, independent of the values of other parameters. Finally, the parameter $\beta = Yl$ captures the degree of leverage and thus the importance of limited liability. The larger the value of β (i. e., the higher the leverage l of traders), the more traders rely on external financing and the more prominent the effect of limited liability becomes. Hence, we obtain the result that the emergence of bubbles may become possible in the context of a high degree of leverage.

General Price Paths. We have demonstrated that the asset market can exhibit a bubble if both (11) and (12) hold by constructing an example bubble, the trinomial bubble. We now show that these conditions are not not only sufficient, but also necessary for bubbles to exist, such that Theorem 1 is complete. Traders are not willing to buy the risky asset if its expected return is lower than that of storage. That is, if there is no value of ϕ that lets the expected return of the asset rise above $Y - \beta$, bubbles will not be possible. This, in turn, is the case if condition (12) is violated: its right-hand side represents the highest expected possible return of the asset (that is, the highest point of the curve in Figure 3), which needs to be above the payoff of storage. This argument does not hinge on the trinomial price path. Instead, it is valid for any distribution of probability mass across different values of ϕ. Condition (12) is therefore necessary for traders to buy the risky asset and thus for bubbles to exist for general price paths. Regarding condition (11), we have already shown that it is necessary: the curve will be below the line in Figure 3 if the condition does not hold. Put differently, storage is again the dominant investment strategy for all ϕ and bubbles cannot exist. To sum up, for a given set of parameters, if no trinomial bubble path exists, no bubble can exist at all. However, if a bubble does exist, its path depends on the evolution of price expectations; it cannot be unique.

4 Bubbles – Household Participation

We have analyzed the condition under which traders are willing to invest in a bubble. Traders, however, borrow from households. If households know that traders might invest their money into overpriced assets, will they lend in the first place? We have assumed that the risk-free return is Y for traders but only $\lambda Y < Y$ for households. If there was no risky asset, households would always invest through traders. In the presence of the risky asset, however, there are several reasons why households might not do so.

First, even in the absence of a bubble, households anticipate (and dislike) the fact that traders invest in the risky asset, since they lose their investment if the underlying firm goes bankrupt. The probability that the commissioned trader buys the asset depends on the asset's price and thus on its market capitalization. If the asset is expensive, it soaks up a lot of funds and this probability is large. The willingness to lend also depends on households' expectations regarding market depth N. More uncertainty about N also implies a higher expected N. For a given price, market capitalization is then smaller compared to total wealth and the probability that a trader invests in the asset is lower. Inequality (10) states the conditions under which households lend to traders in steady state.

Second, within a bubble, households are even more reluctant to lend because they suffer from a bursting bubble. This implies that households might be willing to lend in the absence of a bubble but not in the presence of a bubble. As the bubble evolves, the probability of a burst increases and households become more reluctant to invest. If households reduce their lending or stop lending completely at some point because their participation constraint becomes binding, the bubble bursts. By backward induction, the bubble can then not emerge in the first place. We can show that, given all other parameters, a critical $\bar{\lambda} > 0$ exists such that for any $\lambda \leq \bar{\lambda}$ the participation constraint of households does not constrain any bubble equilibrium.

The Example Bubble. Consider the case of the example bubble with a trinomial price path. The probability that the trader invests in the asset is

$$\int_{p_t}^{\infty} \frac{p_t}{N} f(N)\,\mathrm{d}N = \int_{p_t}^{\infty} \frac{p_t}{N}\frac{\gamma}{p_t}\left(\frac{N}{p_t}\right)^{-\gamma-1}\mathrm{d}N = \frac{\gamma}{\gamma+1}. \tag{21}$$

Otherwise, with probability $1/(\gamma+1)$, the trader purchases the safe asset and the household's return is β. In case the money is invested in the asset, there are the three cases of the trinomial bubble tree (13). First, the firm may default with probability $1-q$, in which case the household gets nothing and pays the verification cost c. Second, given that the trader has bought the asset, the bubble continues with probability $Q_t = q\,p_t^{\gamma+1}/p_{t+1}^{\gamma+1}$ and the household receives β. Third, the bubble bursts and the price drops to the steady state $\bar{p}$. The corresponding probability, conditional on the fact that the trader has bought the asset, equals $Q_t = q\,(1-p_t^{\gamma+1}/p_{t+1}^{\gamma+1})$. Households whose trader has invested in the risky asset get a fraction $\bar{p}/p_t$ and pay the verification cost c. Summing up, the expected return to a household is

$$\frac{1}{\gamma+1}\,\beta + \frac{\gamma}{\gamma+1}\left[(1-q)\,(-c) + q\,\frac{p_t^{\gamma+1}}{p_{t+1}^{\gamma+1}}\,\beta + q\left(1-\frac{p_t^{\gamma+1}}{p_{t+1}^{\gamma+1}}\right)\left(\frac{\bar{p}}{p_t}-c\right)\right] \tag{22}$$

This term has to exceed $\lambda Y l$. We can hence calculate a critical $\bar{\lambda}$. If λ is above this critical point, the gains from trade are too small; households are better off buying the safe asset themselves instead of lending to traders. The critical $\bar{\lambda}$ depends on p_t and $p_{t+1} = \phi_t\,p_t$. As the bubble evolves, both p_t and ϕ_t increase. Both effects reduce households' expected return (22). Consequently, $\bar{\lambda}$ decreases and households might become unwilling to lend at a certain stage: the expected negative return from a potential investment in the bubbly asset can no longer be compensated with possible interest payments from investment in storage. To derive the corresponding condition, we must verify whether households are willing to invest even at arbitrarily high p_t. The last addend in the brackets in (22) vanishes and we can set ϕ_t to the limit ϕ. Expression (22) becomes

$$g(\phi) = \frac{1}{\gamma+1}\left[(c+\beta)\frac{q\,\gamma}{\phi^{1+\gamma}} + \beta - c\,\gamma\right].$$

This value has to exceed $\lambda\,Y l$. Solving for λ, a bubble can thus only evolve if

$$\lambda \le \bar{\lambda} := \frac{1}{\gamma+1}\,\frac{1}{Yl}\left[(c+\beta)\frac{q\,\gamma}{\phi^{1+\gamma}} + \beta - c\,\gamma\right]. \tag{23}$$

with ϕ implicitly defined by (20): $Y-\beta = q\,\phi^{-\gamma}\,(\phi-\beta)$. This condition is depicted as a thick red curve in Figure 4. Note first that (23) is stricter than (10), the household participation constraint in the absence of a bubble. Moreover, if (12) fails to hold, then (20) has no solution for ϕ and bubbles are impossible because traders do not invest in the asset. If (12) holds and a $\phi > 1$ is well-defined, households participate according to (23) only if lending to traders beats the opportunity investment.[22]

General Price Paths. In the proof of the following theorem, we show that condition (23), stemming from the households' participation constraint, holds not only for our example bubble with a trinomial price process, but also for bubbles in general.

Theorem 2 *Assume that* (11) *and* (12) *hold, such that a* $\phi{>}1$ *exists with*

$$Y-\beta = q\,\phi^{-\gamma-1}\,(\phi-\beta).$$

Then if

$$\lambda < \bar{\lambda} := \frac{1}{\gamma+1}\,\frac{1}{Y\,l}\left[(c+\beta)\frac{q\,\gamma}{\phi^{1+\gamma}} + \beta - c\,\gamma\right], \tag{24}$$

households' participation constraint does not restrict the existence of bubbles. If this condition does not hold, bubbles are impossible because households would stop lending to traders at some point.

In sum, the households' participation constraint reduces the parameter space in which bubbles are possible but does not prohibit the possible existence of bubbles. There is a critical value $\bar{\lambda}$ such that for smaller λ households participate even in fully-grown bubbles. For larger λ, households would not participate; hence, bubbles cannot emerge.

We now have two conditions. If equations (11) and (12) hold, traders invest in bubbles. Condition (24) then determines whether households lend to traders in the presence of bubbles. If it fails to hold, we are in the steady state ($\phi = 1$) and this condition reduces

[22] If households do not lend to traders, the price of the risky asset will adjust to the its fundamental value. Traders are then again indifferent between storage and the risky asset. No bubbles will occur.

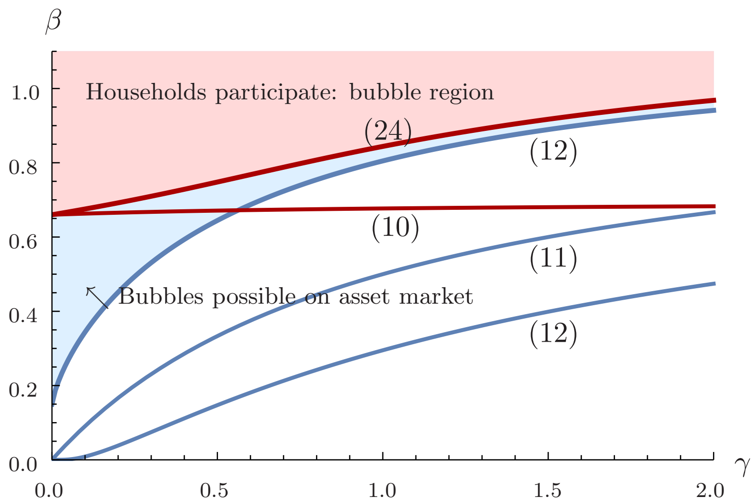

Figure 4: Parameter Range where Bubbles are Possible

The figure is based on a numerical example with $Y = 1.1$ and $q = 0.95$ (as in Figure 1) and additionally $\lambda = 2/3$, $e = 0.1$, and $c \to 0$. The blue curves are thus the same as in Figure 1. Condition (10) holds above the thin red curve (which nearly looks like a horizontal line). Here, households lend to traders in the absence of a bubble. Condition (24) holds above the thick red curve: households lend to traders even if they know there is an asset bubble. The thick red curve is always above the thick blue curve, which means that (24) is stricter than (12). In the red shaded region, bubbles are possible.

to (10), determining whether households lend at all. How does the new condition (24) depend on exogenous parameters? Figure 4 shows one general property. Exogenous parameter constellations that tend to invalidate condition (12) also tend to invalidate (24). In particular, households are less willing to lend in a bubble when the underlying asset is risky (low q, not in the picture), when the trader's leverage ratio is low (low $\beta = Yl$, for a given Y) or when uncertainty about market depth N is low (high γ). The only difference to the comparative statics for Theorem 1 is the effect of the opportunity yield Y; households are willing to lend to traders in particular when Y is high (not in the picture). Two of these statements deserve additional explanations. First, why is a household reluctant to lend to a trader with a low leverage ratio? If leverage is sufficiently low, traders shun investment in an overpriced asset. If a bubble exists nevertheless, its expected price path must be steep enough to compensate traders. This increases the risk of a burst and hence worsens the prospects for households. Note that although a lower leverage level implies a lower β, that is a lower return in case the bubble does not burst, the return per invested dollar β/l remains unchanged. Yet, since monitoring costs c do not depend on the amount of money invested, they become larger in relative terms. A lower l hence reduces the expected payoff from lending to traders in addition to the increased riskiness. Second, why is the effect of Y positive? Y has again a double role. On the one hand, the effect of a higher Y is detrimental for households because with a more profitable safe investment, the bubble has to grow faster to achieve market clearing. It is thus again more likely to burst. On the other hand, as in the case of l, we obtain an additional effect for $c > 0$. While a higher Y increases the return from investing through traders in case the bubble continues (as $\beta = Y\, l$) and households' outside option proportionally, the monitoring costs stay the same. They hence fall in relative terms, which makes lending to traders more attractive.

Proposition 2 *If the household participation constraint* (24) *holds for given values of* Y, q, γ, l, λ, *and* c, *then it also holds for any lower values of* γ, λ, *and* c. *It also holds for any larger values of* q, l *and* Y.

In addition to the new effects of λ and c, variations in the parameters of the model provide the same conclusions as in Proposition 1, except for Y. This implies that, for a given Y, the same parameter constellations that induce households to invest through traders make bubbles possible. It is therefore likely that in situations in which households find it optimal to invest via traders, the resulting leverage creates the conditions for bubbles. The optimal investment strategy of households hence ultimately reduces their expected payoffs, as households do not take the externalities of their investment decisions on equilibrium prices into account.

5 Welfare and Policy Measures

In this section, we examine the effectiveness of policy measures that have been suggested to prevent the emergence of bubbles. Specifically, we look at a financial transaction (Tobin) tax and capital requirements. For each policy measure, we do two things. First, we check whether the policy prevents bubbles. Second, we analyze the measure's effect on the steady state. As bubbles per definition imply multiple equilibrium price paths, it is inherently impossible to analyze how a certain policy influences welfare if a bubble is not prevented. We can, however, make some statements about relative welfare in situations with and without a bubble. A measure that prohibits a bubble always lowers the utility of the initial owners of the asset, because they sell it at the steady-state price instead of an inflated price. Similarly, policies that reduce the steady-state price will also harm the initial generation. On the other hand, we will show that future generations of households can benefit from policy interventions. Calculating an aggregate effect would require additional assumptions on the discount rate between generations of households and traders.[23] We therefore restrict ourselves to analyzing the inter-generational distributional consequences of bubbles, which can be derived without any further assumptions. When evaluating welfare of the initial and future generations, here and in the following, we will always look at *realized* welfare of the initial generation and *expected* welfare of future generations.

[23] For example, assuming an intergenerational discount factor of $\bar{\rho}$ as well as $Y\,\bar{\rho} > 1$ would give a negative aggregate welfare effect of bubbles. If $\bar{\rho} \to 0$ and only the initial generation counted, bubbles would increase welfare.

5.1 Welfare

Welfare of a single generation is given by the sum of expected consumption of all households and of all traders of the same generation. Here, we assume that one dollar spent on consumption by traders generates the same welfare effect as one dollar spent by households. First, let us consider whole generations, i. e., traders and households jointly. Expected welfare depends on the relative investment into storage and the risky asset. Storage delivers a safe return, while the asset might be sold at a higher or lower price to the next generation. In equilibrium, both yield the same expected return to traders due to arbitrage. Remember, however, that traders' arbitrage condition ensures that their expected *private* returns align across both assets. Given that traders take their limited liability into account when investing, they engage in too much risk, seen from an aggregate perspective. That is, traders drive the price of the risky asset up, such that its expected *social* return (i. e., under full liability) is below the return to storage; in fact, it is lower than one. The bigger a bubble gets the larger will be this effect, as the investment decision becomes riskier and, additionally, the bubble crowds out investment in the safe (and productive) asset. Welfare of a generation as a whole is thus negatively affected by bubbles. Behind this welfare loss is the fact that the bubble does not produce any output (besides dividends); it just pulls consumption forward from future generations. This becomes more and more risky, as the asset's price might drop while in possession of the respective generation. Moreover, welfare is furthermore reduced by the monitoring costs. These costs, which represent a deadweight loss, occur in steady state only if the underlying firm goes bankrupt. In a bubble they are also paid if a bubble bursts, such that their value increases in expectations. Realized welfare of the initial generation, on the other hand, depends positively on the existence of a bubble. As it has already made its investments, a bubble that increases the resale value of the asset can only have beneficial effects.

To disentangle the effects on each group of agents, let us look at traders and households in isolation. Consider the expected utility differences between a situation with a bubble and one without. The initial generation of traders gains unambiguously from a bubble, as expectations about future prices move the current price p_0 above the steady state. All following generations of traders must be indifferent between buying the asset and investing in storage (condition 17). Traders are indifferent between these two options also in steady state. As the return to storage is independent from the existence of a bubble, it follows that, for traders, the expected return of the asset is the same with and without a bubble. Expected welfare of future generations of traders is hence unchanged by bubbles.

Expected utility of households, on the other hand, is affected negatively by a bubble. In fact, households cannot gain from a bubble: if the bubble continues, they obtain β. As soon as the price deviation from the steady state is large enough, they lose part of their investment when the bubble bursts. In addition, the higher the current price, the more money is absorbed by the bubble and the more likely traders are to invest in the bubbly asset. The underlying reason that generates this asymmetric distribution of gains and losses in expected welfare between traders and households is rooted in the limited liability of traders. Since traders default in case of bankruptcy, parts of the risk of a bursting bubble are shifted to households. Households recognize this, but as long as their own investment opportunities are sufficiently inferior relative to those of traders (condition 24), they still expect a higher return from investing through traders. Note that the expected welfare losses of households are proportional to the degree of risk-shifting. The higher the level of limited liability (higher l), the higher the risk that traders are willing to take, which is then partly shifted to households. Limited liability hence leads to socially suboptimal level of risk-taking, imposing a negative externality in form of a lower expected return on households. Without limited liability ($\beta = 0$), incentives of traders align with those of households, i. e., traders fully internalize the risks arising from bubbles. Social and private returns align and no welfare loss occurs. Because of the same reason, however, bubbles cannot occur in the first place and assets would trade at their fundamental value.

If households had heterogeneous outside options, there can be additional welfare costs, reinforcing the negative effects of risk shifting. Those households with a better alternative investment possibility stop lending to traders if the bubble has grown large (and risky) enough. Since households know about the risk that traders might invest in the risky asset, only those households with a low outside option continue to lend. The high-outside-option households prefer to invest in their alternative investment technology. Because traders have access to a superior technology, this constitutes a welfare cost comparable to Conlon (2015). Appendix A analyzes this extension in detail.

Proposition 3 *For given parameter values that allow for bubbles, utility of the initial generation of traders is lowest in steady state and higher on a bubble price path. For all future generations of traders, expected utility is the same on all price paths. For all generations of households, expected utility is highest on the steady-state price path.*

Note the difference from models of rational bubbles in overlapping-generations models with a mechanism similar to Tirole (1985). In these models, bubbles are possible if the economy features dynamic inefficiencies. Investing less in capital (and more in bubbly assets) increases current and future consumption, such that bubbles enhance welfare. In our model, bubbles can exist even though the rate of return on productive investments exceeds the growth rate of the economy (which is zero) because of unknown market depth and limited liability.

Moreover, the generation that holds the risky asset at the time of a bursting bubble has a lower utility ex post than in steady-state. This is true for traders, who have invested in an asset that underperforms storage, and households that obtain less than the contracted amount from traders. As agents are aware of this fact, they would oppose policy measures that pop the bubble after they have bought but not yet sold the asset. Furthermore, if no bubble has emerged so far, the current asset holders would benefit from the emergence of a bubble. Hence, at no point in time would a majority of agents currently owning the risky asset and those about to buy it be in favor of policy measures that prevent or burst bubbles.

5.2 Financial Transaction Tax

Financial transaction taxes (FTTs) have been discussed in the theoretical literature at least since Tobin (1978) prominently proposed such a tax. Scheinkman and Xiong (2003) show that a FTT affects prices only to a limited degree, but can change transaction volumes strongly. Also Constantinides (1986) finds limited price effects. More recent contributions include Adam, Beutel, Marcet, and Merkel (2015), who argue that while such a tax reduces the size and length of boom-bust cycles, it simultaneously increases the likelihood of these cycles. In our model, a period is interpreted as the investment duration of a trader. A tax on buying the asset, or on storage, can thus be interpreted as a FTT. Due to the setup, all assets are traded each period. The effects of a transaction tax in our model do therefore not work via a reduction in the volume of transactions but only via price effects, as would, e. g., a property tax on the asset. In the following, we assume without loss of generality that the tax must be paid by the buyer of an asset. We call τ the tax rate on transactions of the safe asset and τ' the (potentially different) tax rate on the risky asset. Tax revenues are redistributed as lump-sum transfers to households of the same generation that has paid the taxes, after investment decisions have taken place. Let us discuss how equation (20) changes. A buying trader has one dollar to spend. As the price of storage is normalized to unity, the gross price including the tax is $1+\tau$. The trader can hence afford to buy $1/(1+\tau)$ units of storage. After one period, the return of a trader after repayment of the debt to households is $Y/(1+\tau)-\beta$. In this context, it is important to remember that β is endogenously determined by the negotiations between these parties. Given that the outside option of traders falls with a rising τ, the scheduled repayment to the households β falls as well. Specifically, β results as $Yl/(1+\tau)$. For the risky asset, the argument is similar. If the current price is p_t, the trader can buy $1/(p_t\,(1+\tau'))$ units. Like in the discussion of (20), we can concentrate on high prices, such that the dividend d becomes irrelevant. After repayment of the debt, the expected traders' return is $q\,\phi^{-\gamma-1}\left(\phi/(1+\tau')-\beta\right)$.[24] Under such a tax regime, the market-clearing condition (20), observing the endogeneity of β as it depends on τ, changes to

$$\frac{Y(1-l)}{1+\tau} = q\,\phi^{-\gamma-1}\left(\frac{\phi}{1+\tau'} - \frac{Yl}{1+\tau}\right).$$

Following the same procedure as for (11) and (12) in Theorem 1, we derive the modified conditions for the existence of bubbles as

$$1 < \frac{\gamma+1}{\gamma}\frac{1+\tau'}{1+\tau}Yl, \tag{25}$$

$$Y^{\gamma+1}(1-l)l^{\gamma} \leq q\,\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\left(\frac{1+\tau}{1+\tau'}\right)^{\gamma+1}. \tag{26}$$

[24] Here we use the trinomial price path of Section 3.1. Following the arguments in Section 3.1 and observing that the expected payoff from investing in a bubble with a FTT remains concave in ϕ shows that if trinomial bubbles are not possible with a FTT, they are neither possible for any other expected price path. A similar argument holds for Section 5.3.

As visible in these equations, an increase in τ' has the same effect as an increasing Y. We know from Proposition 1 that this can destroy the possibility of bubbles. Intuitively, a tax reduces the asset's return for a given amount of risk. If the tax is high enough, traders prefer storage instead. Yet, the way in which the tax is implemented is crucial. A tax on storage displays the exact opposite effects as one on the risky asset and can therefore create the possibility of bubbles. In particular, it reduces the return from storage and, simultaneously and proportionally, the equilibrium value of β. For a given tax on the risky asset, the latter becomes relatively more attractive. Additionally, a tax on the safe asset changes the households' participation constraint (24). Households obtain a lower outside option and a proportionally reduced payment from traders. In case traders default, however, households have to pay the same monitoring costs as before. A high tax can therefore deter households from lending to traders. We therefore obtain the result that a tax on storage can either create the possibility of bubbles or destroy the market for intermediation.

If the tax is levied on all financial assets, including storage ($\tau = \tau'$), it only impacts households' participation constraint. That is, a high enough tax rate can eliminate the possibility of bubbles, but again only at the cost of a breakdown of the intermediation market.

Proposition 4 *If a financial transaction tax is levied on the risky asset only, the range of parameters in which bubbles are possible is reduced. If it is levied on the safe asset, this range increases or households might stop lending to traders. If it is levied with equal rates on both assets simultaneously, households might stop lending to traders.*

In practice, identifying assets that correspond to the safe or the risky asset of the model is difficult. This can constitute a major obstacle to the implementation of a Tobin tax that aims at preventing bubbles. The effects of a FTT on welfare are further complicated by the fact that the tax does not only affect the possibility of bubbles, but changes steady-state prices as well. As shown in the proof of the next proposition, however, steady-state consumption actually increases with higher transaction taxes on the risky asset. This is remarkable, as the collected taxes are just returned to the households without being invested at all. The underlying reason is that such a tax fulfills exactly the function that the households cannot carry out: it penalizes investment in the risky asset. Since it is a transaction tax, it taxes the purchase of the risky asset independently of its return. It hence becomes unattractive to invest in the risky asset for traders, such that overall investment in storage actually rises with higher tax rates. The opposite effect obtains if the tax is levied on the safe asset: storage is less attractive and expected steady-state consumption falls. Steady-state consumption also falls if the tax is levied on both assets with the same rate.

We know from Proposition 3 that bubbles reduce expected welfare, except for the initial generation. Adding the result that a correctly implemented Tobin tax lowers the asset's price in steady-state, we can conclude that such a tax reduces realized welfare of the initial generation. On the other hand, the tax increases expected welfare of future generations. A tax on the safe asset, in turn, reduces welfare for future generations for sure if we start in a no-bubble situation, as it decreases steady-state welfare and may create the possibility of bubbles or destroy the intermediation market. Due to an increase of the steady-state price of the risky asset, the initial generation gains from such a tax, as long as the intermediation market does not break down. Traders' arbitrage considerations are not altered by a tax on both assets. Given the assumption that tax revenues are not invested, however, steady-state consumption falls with such a tax, additional to the possible breakdown of the intermediation market.

Proposition 5 *A financial transaction tax on the risky asset that prohibits bubbles increases expected welfare for future generations but reduces welfare of the initial generation. A tax on the safe asset, or on both assets with the same rate, reduces expected welfare for future generations.*

5.3 Capital Requirements

Another policy measure often discussed in connection to financial stability are capital requirements. Morrison and White (2005), Van den Heuvel (2008), and Harris, Opp, and Opp (2015), to name just a few recent contributions, discuss the effects of capital requirements in a variety of settings. In our setup, the analysis of capital requirements is a straightforward exercise, no model modifications are needed. Capital regulation requires that for e dollars of equity, a trader can borrow up to $l \leq 1-e$ dollar. The balance sheet total is thus $e+l$, and the equity ratio equals $e/(e+l)$. If the regulator stipulates stricter capital requirements, the equity ratio must increase; thus, l must decline. Because $\beta = Y\,l$, a smaller l leads to a smaller β. Based on Proposition 1, we know that a smaller β tends to entail a unique rational-expectations equilibrium, i. e., no bubble. We hence obtain the following proposition.

Proposition 6 *If capital requirements are increased, the range of parameters in which bubbles are possible is reduced.*

Recall that, if the price path is too steep, bubbles do not exist because any potential bubble would be too likely to burst. Therefore, traders cannot be compensated for an investment in an overpriced asset by further expectations regarding price increases. If traders are highly leveraged, however, the costs of a burst are shifted to households. Since capital requirements reduce leverage (traders have more "skin in the game") they can eliminate potential bubbles.

Again, we *first* look at the welfare consequences if capital requirements prohibit a bubble. This increases expected welfare of future generations of households and reduces that of the initial generation. *Second*, a decrease in l reduces leverage and hence lowers the steady-state price. This could be positive for future welfare, as more resources get invested into the safe asset. As shown in the proof, however, the price falls more slowly than the available resources for the trader. The relative share of funds flowing into the risky asset, $\bar{p}/(e+l)$, hence increases with higher capital controls. Since the relative share of the safe asset thus falls and traders have absolutely less money to spend, they actually invest less in the safe (and in the risky) asset. The lower steady-state price is also not in the interest of the initial generation. *Third*, related to the previous point, a decrease in l raises the amount that households must invest themselves. Because of the households' investment disadvantage λ, this is detrimental for welfare. The overall effect of a tightening of capital controls thus depends on their initial level. If capital requirements are increased, but not enough to prevent the occurrence of bubbles, no welfare conclusions can be drawn. If capital requirements are increased sufficiently to prevent bubbles, further increases reduce welfare.[25] At some point along the path of increasing capital requirements, households stop lending to traders, as the rising probability that traders invest in risky assets (second point above) makes investments through traders less attractive than investing in storage.

Proposition 7 *Increases in capital requirements beyond the point where bubbles are eliminated reduce welfare for households and traders of all generations.*

6 Conclusion

Our model endogenizes a specific reason why the price of an asset may deviate from its steady-state price. If market depth is unknown, a trader may be willing to spend more on an asset because she expects to earn even more when she sells the asset. This price deviation can occur with unchanged fundamentals. It is completely driven by expectations and is dynamic, typically involving unpredictable abnormal returns until this bubble bursts. Such bubbles can occur especially if traders are highly leveraged and if the information about market depth is imprecise. In addition, due to leverage, also the steady-state price of a risky asset exceeds its fundamental value. This price deviation is not caused by expectations about rising prices, but by traders' risk-loving behavior. It involves no dynamics and is therefore not a bubble, in our definition.

The policy measures differ in their impact on a bubble and the static price deviation in steady state. A correctly implemented Tobin tax brings a welfare improvement in steady state, and it can puncture a bubble. Capital requirements bring welfare deteriorations in the steady state, but can puncture bubbles as well. By virtue of its relative simplicity, the model lends itself to discussions of related phenomena. For example, one could consider multiple assets and discuss whether the collapse of a bubble in one market can be contagious for other markets. One could also introduce this kind of bubbles to macro models and investigate its effects on business cycles and growth. Especially after the recent bursts of housing bubbles, applications seem both numerous and relevant.

[25] It is not possible to make definitive statements about welfare in a situation with capital requirements just high enough to eliminate bubbles relative to the situation without any policy intervention. This is due to the effect of the policy intervention on the steady-state price. Proposition 3 establishes that welfare in a bubble is lower if compared to a situation with the same parameter values but no bubble, except for the initial generation of a bubble. Put differently, we cannot calculate welfare in a bubble but we know that it is below the corresponding steady-state situation for future generations. Increases in capital requirements can eliminate the possibility of bubbles, but have at that point already reduced welfare in the alternative scenario of a steady state. Proposition 3 does therefore not apply and no definite statement about relative welfare measures can be made.

A Welfare with heterogeneous households

This appendix develops a version of the model with two types of households, which differ in their alternative investment possibility. If this is the case, it can happen that one type of households stops lending to traders if the bubble becomes too large. Specifically, since households cannot observe actions of traders and a large bubble increases the probability that the commissioned trader invests in the risky asset, households with a high outside investment possibility might prefer to use this alternative instead of investing through traders. The market hence resembles a classical lemons market. Because traders have access to a superior investment technology, the breakdown of this part of the market constitutes a welfare loss.

In this version of the model, there is a discrete point when one type of households drops out. For demonstrative purposes we choose an illustrative case and assume parameter values such that this will happen in the initial period when the bubble emerges.[26]

A.1 Changes to baseline model

There are two types of households, indexed by $i \in \{L, H\}$. The types differ in their alternative investment disadvantage λ^i, where $\lambda^L < \lambda^H$. The share of households of type L in the economy is x, with $0<x<1$. As will be shown below, only these households will continue to lend to traders in a bubble. The variable x' denotes the share of households that currently participate. It is either 1 (all households participate), x (only those households with λ^L participate) or zero (nobody participates). The relevant liquidity in the market for the risky asset is therefore $x'N$, as $x'N$ households each lend $1{-}e$ dollars to $x'N$ traders, each of which with e dollars endowment.[27]

The conditional probability that there is enough liquidity in the market to sustain p_{t+1}, given that p_t was already reached, is the same for $x'N$ as for N. Given that the share x' of households participates, the minimum N can be deducted from the price as $N_0 = p_t/x'$. We hence get the same relevant distributions as for $x' = 1$, our baseline case, and the arbitrage condition for traders (20) remains unchanged. We can therefore concentrate on the modified participation constraint of households. Note that also the probability that the commissioned trader invests into the risky asset remains unaffected,

$$\int_{p_t/x'}^{\infty} \frac{p_t}{Nx'} f(N)\, \mathrm{d}N = \int_{p_t/x'}^{\infty} \frac{p_t}{Nx'} \frac{\gamma}{p_t/x'} \left(\frac{N}{p_t/x'}\right)^{-\gamma-1} \mathrm{d}N = \frac{\gamma}{\gamma+1}.$$

[26]We therefore do not have to worry about a break in the continuation probability of a bubble that would be there if households dropped out at a later point in time. There is only one case which is special: if a relatively large share of households decides not to lend to traders anymore in the very first period of a bubble, such that the amount of liquidity in the market for the risky asset is not even enough to support the initial price increase. We would then obtain a one-period bubble. We ignore this case, as it does not add to the intuition for this model version.

[27]More precisely, unleveraged traders invest their endowment e if $x' = 0$. However, the asset trades at its fundamental value and bubbles are not possible in this case. For $x' > 0$, leveraged traders push the asset price above the fundamental value, such that unleveraged traders prefer storage.

The expected payoff for households that lend to traders, equation (22), is hence the same as in the baseline version. This payoff, however, needs to be above the individual outside option in order for the household to participate,

$$\frac{1}{\gamma+1}\beta + \frac{\gamma}{\gamma+1}\left[(1-q)(-c) + q\frac{p_t^{\gamma+1}}{p_{t+1}^{\gamma+1}}\beta + q\left(1-\frac{p_t^{\gamma+1}}{p_{t+1}^{\gamma+1}}\right)\left(\frac{\bar{p}}{p_t}-c\right)\right] > \lambda^i Yl.$$

Looking at the limit case of $p_t \to \infty$ yields the same condition as before, see equation (24),

$$\lambda^i \leq \frac{1}{\gamma+1}\frac{1}{Yl}\left[(c+\beta)\frac{q\,\gamma}{\phi^{\gamma+1}} + \beta - c\,\gamma\right], \quad i \in \{L, H\}.$$

Let us assume that the following two conditions hold

$$\lambda^L < \frac{1}{\gamma+1}\frac{1}{Yl}\left[(c+\beta)\frac{q\,\gamma}{\phi^{\gamma+1}} + \beta - c\,\gamma\right] < \lambda^H.$$

In a bubble, the low-type households hence continue to lend, while the high types drop out and invest in their alternative, inferior technology. To guarantee that all households lend to traders in the steady state, we need to assume that the expected return of investing through traders is also high enough for the high types (insert λ^H into the steady-state participation constraint of households 10 instead of λ).

A.2 Welfare

Steady-state welfare is the same in the heterogeneous-households setup as in the baseline model, since all households lend to traders in steady state. The same is true for the initial period when the bubble emerges, as households have decided to invest through traders at the steady-state price. We therefore only need to compare welfare in a bubble in both versions of the model. In the heterogeneous-households setup with only one type of households lending to traders in a bubble, expected welfare (consumption) of a whole generation in period t amounts to

$$\begin{aligned} E_tW_t &= E_t xN\left[\frac{p_t}{xN}Q_t\frac{p_{t+1}+d}{p_t} + \frac{p_t}{xN}(q-Q_t)\frac{\bar{p}+d}{p_t} + \left(1-\frac{p_t}{xN}\right)Y - \frac{p_t}{xN}(1-Q_t)c\right] + E_t(1-x)N\lambda^H Y \\ &= p_tQ_t\frac{p_{t+1}}{p_t} + p_t(q-Q_t)\frac{\bar{p}+d}{p_t} + E_t\left[xN(1-\lambda^H) + N\lambda^H - p_t\right]Y - p_t(1-Q_t)c. \end{aligned}$$

Subtracting the expression for welfare of one generation in a bubble in the baseline version, the sum of equations (36) and (38), yields

$$E_0(1-\lambda^H)N(x-1) \leq 0,$$

which is negative as long as those households with a high outside investment alternative still have a disadvantage against traders ($\lambda^H < 1$) and at least some households use this alternative in a bubble ($x < 1$). We hence have an additional welfare loss from those households that stop investing through traders if a bubble emerges.

B Proofs

Proof of Lemma 1. Due to asset-market clearing, the risky asset is always bought by some trader. Hence, the traders' investment return R is inherently risky. The contract between trader and household contains the trader's investment, the household's lending, a repayment function $z(R)$ and a function $B(z) \in \{0;1\}$ with $B(z)=1$ if the household verifies the true return R after some specific repayment z, and $B(z)=0$ if it does not. Then the optimal contract must fulfill incentive compatibility,

$$z(R_1) < z(R_2) \implies B(z(R_1)) = 1, \tag{27}$$

for all $R_1, R_2 \geq 0$. If the trader pays less in some state and more in another, she must be controlled in the state where she pays less. Otherwise, she would prefer to repay the lower amount also in the better state. Of course, $z(R) \leq R$ must hold for all $R \geq 0$. The trader cannot repay more than she has earned. The trader's participation constraint is

$$E[R - z(R)] = E[R] - E[z(R)] \geq e\,Y.$$

The investment returns R, she repays $z(R)$. The opportunity investment of e would have returned $e\,Y$ (either via investment in storage or the risky asset, see footnote 14). Since the household has the bargaining power, the above will be an equality, $E[z(R)] = E[R] - e\,Y$. Finally, the household maximizes

$$E[z(R)] - c\,E[B(z(R))] = E[R] - e\,Y - c\,E[B(z(R))].$$

This implies that the household wants to minimize the probability that it needs to verify. Because of (27), the verification states must be the ones with the lowest return R. In all other states, the repayment has to be the same (some value β that is determined in the bargaining process between household and trader). Also, in the verification states, the trader repays all she has ($z(R) = R$). Otherwise, the household would benefit from increasing the repayment in some states to R and compensating by reducing β, which also lowers the probability of verification. Putting the arguments together, we receive $z(R) = \min\{\beta; R\}$. In our setting, in comparison to Gale and Hellwig (1985), there is one additional reason *not* to have an standard debt contract (SDC). With an SDC, the trader is incentivized to invest in the risky asset, with a positive probability of default and a negative externality on the household. A non-SDC, however, entails a verification cost of c with probability 1, whereas an SDC requires c only if the trader invests in the risky asset *and* the asset defaults. Therefore, if c is high enough, an SDC is the dominant financial contract, i. e., the results of Gale and Hellwig (1985) hold. ■

Proof of Theorem 1. In the exposition in the main text, we have already treated the case in which traders are not paid if a bubble bursts. Hence, we begin the proof of the theorem by providing a condition for this case and analyzing the alternative. If a bubble bursts without the firm going bankrupt, the firm still pays the dividend. The return of the trader is then

$$\max\left\{\frac{d}{p_t}+\frac{\overline{p}}{p_t}-\beta;0\right\}=\max\left\{\frac{d+\frac{d\,q}{Y-q-\beta\,(1-q)}}{p_t}-\beta;0\right\}.$$

This equation implies that if the price is only slightly above the steady-state price $\overline{p}$ (i. e., the bubble is small), the trader receives a payment even when the bubble bursts. The corresponding condition is

$$p_t<\check{p}:=\Big(d+\frac{d\,q}{Y-q-\beta\,(1-q)}\Big)/\beta. \tag{28}$$

If p_t is less than $\check{p}$ such that (28) is satisfied, a modified version of (17) applies. In the market equilibrium,

$$\begin{aligned} & Y-\beta=Q_t\left((p_{t+1}+d)/p_t-\beta\right)+(q-Q_t)\left((\overline{p}+d)/p_t-\beta\right)\\ \iff\quad & \frac{Y-\beta}{q}=\Big(\frac{p_t}{p_{t+1}}\Big)^{\gamma+1}\frac{p_{t+1}}{p_t}+\Big(1-\Big(\frac{p_t}{p_{t+1}}\Big)^{\gamma+1}\Big)\left(\frac{\overline{p}}{p_t}+\frac{d}{p_t}-\beta\right). \end{aligned} \tag{29}$$

Again, beginning from p_t, we have an implicit equation for p_{t+1} in a rational-expectations equilibrium. Substituting $p_{t+1}=\phi_t\,p_t$, we obtain

$$\frac{Y-\beta}{q}=\phi_t^{-\gamma}+\Big(1-\phi_t^{-\gamma-1}\Big)\left(\frac{\overline{p}}{p_t}+\frac{d}{p_t}-\beta\right).$$

However, in a bubble, the price p_t increases over time and eventually exceeds the threshold $\check{p}$. Therefore, to determine whether bubbles are feasible it suffices to consider the case $p_t>\check{p}$, as done in the main text. Define the right-hand side of equation (17) as

$$\hat{h}(\phi,p)=q\,\min(1,\phi^{-\gamma-1})\,\max\left(0,\phi+\frac{d}{p}-\beta\right). \tag{30}$$

For $\phi>1$ and $\phi+\frac{d}{p}-\beta>0$, this equation turns into

$$h(\phi,p)=q\,\phi^{-\gamma-1}\left(\phi+\frac{d}{p}-\beta\right).$$

In Figure 3, h is plotted as a dotted curve. The key question is whether bubbles are possible for arbitrarily large prices. We are therefore interested in $h(\phi):=h(\phi,\infty)$ with $d/p\to 0$, slightly abusing notation. This gives the solid curve in Figure 3. The derivative of $h(\phi)$ w. r. t. ϕ is

$$h'(\phi)=q\,\phi^{-\gamma-2}\,[\beta\,(\gamma+1)-\gamma\,\phi].$$

The function is strictly concave with a maximum at $\phi^*=\beta\,(1+1/\gamma)$. Thus, $\hat{h}(\phi,\infty)$ is weakly concave, with a maximum at the same $\phi^*=\beta\,(1+1/\gamma)$ if this is greater than one, and with a maximum at one if $\phi^*\leq 1$. If $\phi^*\leq 1$, the maximum rate of return to the trader from the risky asset is $\hat{h}(1,\infty)=q\,(1-\beta)$, which is less than the return $Y-\beta$ on storage since $Y>1$. Thus, bubbles are only possible if $\phi^*>1$, which yields condition (11) in the main text. Intuitively, $\phi < 1$ implies a lower return for the trader without an offsetting decrease in risk, as the market depth ceiling will not be reached for sure for any $\phi \leq 1$.

More generally, condition (17) can only have a solution if the maximum of its right-hand side, i. e., $\hat{h}(\phi, p)$, is above its left-hand side. That is, the maximum expected gross return of the asset needs to weakly dominate storage. For any possible price path, this maximum return for $p \to \infty$ is bounded from above by

$$E\hat{h}(\phi, \infty) \leq E\hat{h}(\phi^*, \infty) = q \, \frac{\gamma^\gamma}{\beta^\gamma \, (\gamma + 1)^{\gamma+1}},$$

where ϕ is now a random variable. Hence, traders will not invest in the risky asset if $Y - \beta$ is larger than the right-hand side of this inequality, giving condition (12). The above demonstrates that conditions (11) and (12) are necessary for a bubble to exist. The trinomial example then shows that these conditions are also sufficient for a bubble, assuming the distribution of N is Pareto. If the distribution is not Pareto, then, to get a sufficient condition, the weak inequality in (12) must be replaced by a strict inequality. ■

Proof of Proposition 1. There are two conditions that can become tighter or looser when changing a parameter: conditions (11) and (12). However, (11) is never binding. This can be seen in Figure 1, where the bubble region is bordered by condition (12) only. Mathematically, the same can be shown by inserting (11), holding with equality, into (12). We obtain $Y \leq (q+\gamma)/(1+\gamma)$, which is a contradiction. Without the additional condition (11), the area below the lower solution of (12) in the figure would also be part of the bubble region. Hence, (11) determines the bubble region, but it does not touch it. Consequently, we only need to check whether (12) becomes tighter in the relevant region when changing a parameter. Condition (12) can be rewritten as

$$(Y - \beta) \, (\gamma + 1)^{\gamma+1} \left(\frac{\beta}{\gamma}\right)^\gamma - q \leq 0. \tag{31}$$

The first comparative static is obvious. The derivative of the left-hand side w. r. t. q is negative, such that the condition is relaxed for higher q. The partial derivative w. r. t. Y is positive; hence, (12) gets tighter as Y increases. The derivative w. r. t. γ is

$$(Y - \beta) \, (\gamma + 1)^{\gamma+1} \left(\frac{\beta}{\gamma}\right)^\gamma \ln \frac{\beta \, (\gamma + 1)}{\gamma}.$$

Condition (11) implies that $\beta \, (\gamma{+}1) > \gamma$. The above logarithm and the complete derivative is thus positive. A larger γ tightens the condition for a bubble. Finally, we want to show that an increase in β relaxes condition (12). We first show that the left-hand side of (31) is concave in β. It is continuous and the derivative w. r. t. β is

$$(\gamma + 1)^{\gamma+1} \left(\frac{\beta}{\gamma}\right)^\gamma \frac{Y \, \gamma - \beta \, (\gamma + 1)}{\beta}. \tag{32}$$

We are interested in the sign of this derivative especially near the border of the bubble region, i. e., where (31) holds with equality. Substituting (31) into (32) yields

$$\frac{q \, \gamma^{\gamma+1} - \beta^{\gamma+1} \, (\gamma + 1)^{\gamma+1}}{\beta \, \gamma^\gamma}.$$

The numerator is negative because

$$q\,\gamma^{\gamma+1} - \beta^{\gamma+1}\,(\gamma+1)^{\gamma+1} < 0 \iff q < \left(\frac{\beta\,(\gamma+1)}{\gamma}\right)^{\gamma+1}$$

is always true, as the right-hand side exceeds unity as a consequence of (11). Explicitly considering the endogeneity of β by setting $\beta = l\,Y$ (see equation 4), (31) becomes

$$(1-l)Y^{\gamma+1}\,(\gamma+1)^{\gamma+1}\left(\frac{l}{\gamma}\right)^{\gamma} - q \le 0,$$

where Y, as before, enters positively on the left-hand side. Leverage l has the same effect as calculated for β above. ■

Proof of Theorem 2. For trinomial bubbles, we have already derived a necessary and sufficient participation constraint, equation (23). We want to show that the condition remains valid for non-trinomial bubbles. It is clear that if (23) holds, households' participation constraint does not prevent the existence of multiple equilibria: the trinomial bubble is one example of an alternative price path. It remains to be shown that, if (23) fails to hold, households' participation constraint is violated also for any other type of bubble. Hence, we need to show that of all possible bubble paths, the trinomial bubble is the most preferred one by households. Then, if for a certain parameter constellation trinomial bubbles do not exist because of households' participation constraint, households are even more reluctant to invest in a non-trinomial bubble. As in the proof of Theorem 1, the function

$$h(\phi) = q\phi^{-\gamma-1}\,(\phi-\beta)$$

provides the value of the future price for the trader for high p, considering that the bubble might burst. The more advanced a bubble already is, the more reluctant households are to invest; hence, we can concentrate on large prices p. As defined in (**??**), $g(\phi)$ gives the expected return to households in this case. These functions are plotted in Figure 5. The lower, blue curve defines the market clearing condition. In a trinomial bubble, the market clears when the blue curve intersects with $Y-\beta$ (dashed line), i. e., traders are indifferent with respect to investing in the bubbly asset at this point. In a general bubble, the return can assume several values with different probabilities, $h(\phi)$ defines an invariant for the probability distribution of ϕ. For the market to clear, $\mathrm{E}[h(\phi)] = Y-\beta$ must hold. The households' expected return is then $\mathrm{E}[g(\phi)]$. One possible solution is the trinomial bubble, where one single value of ϕ has 100% probability mass. Other solutions might have positive variance. We need to show that for all distributions of ϕ with strictly positive variance, households' expected return falls short of that in the trinomial bubble. We hence need to solve

$$\max \mathrm{E}[g(\phi)] \quad \text{s. t.} \quad \mathrm{E}[h(\phi)] = Y-\beta, \tag{33}$$

where the max operator is taken over all probability distributions of ϕ. We rescale the

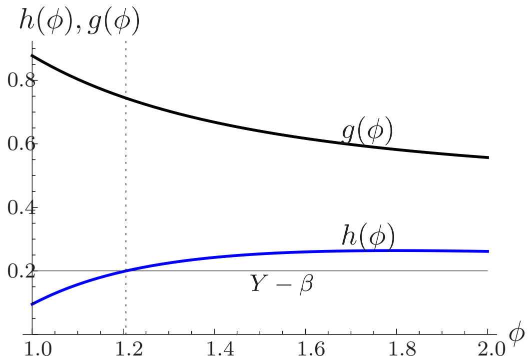

Figure 5: Market clearing condition h and household participation constraint g

Parameters are $\gamma = 1$, $\beta = 0.9$, $q = 95\%$, $d = 1$, $Y = 1.1$, $\lambda = 2/3$, and $c \to 0$.

problem by distorting the ϕ-axis. We substitute $h(\phi) \mapsto x$; thus, $\phi \mapsto h^{-1}(x)$. Problem (33) becomes

$$\max \mathrm{E}[g(h^{-1}(x))] \quad \text{s. t.} \quad \mathrm{E}[x] = Y - \beta, \tag{34}$$

where the max operator is taken over all probability distributions of x. If $g(h^{-1}(x))$ is concave, a mean preserving spread of x deteriorates $\mathrm{E}[g(h^{-1}(x))]$, while $\mathrm{E}[x] = Y - \beta$ still holds. Problem (34) is thus solved by the degenerate one-point distribution. Hence, if we show that $g(h^{-1}(x))$ is concave everywhere, then households prefer trinomial bubbles in which only one ϕ is possible. The implicit function theorem yields

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} g(h^{-1}(x)) = \frac{h'(\phi)\, g''(\phi) - h''(\phi)\, g'(\phi)}{h'(\phi)^3}.$$

Without loss of generality, we can concentrate on distributions of ϕ with support only in the increasing part of $h(\phi)$. If there is probability mass on the decreasing part of h, we can move that mass to the increasing part that has the same level of h. This leaves traders indifferent, but improves the households' expected return since the risk of a bubble burst decreases. Hence $h'(\phi) > 0$ and the denominator $h'(\phi)^3$ is positive. The numerator is

$$-(q\,\gamma)^2\,(c + \beta)\,\phi^{-2(\gamma+2)} < 0. \tag{35}$$

The entire fraction is thus negative. Consequently, $g(h^{-1}(x))$ is always concave and households prefer trinomial bubbles above all other types. If households' participation constraint is violated within the class of trinomial bubbles, it is violated for any bubble.

Finally, we show that in order to establish the conditions under which bubbles can emerge, we can assume that $\beta = Y\,l$ without loss of generality. The relevant question is whether households could have an incentive to increase β to economize on monitoring costs. In a trinomial bubble with a high asset price (which is the relevant case for the existence conditions), the household monitors if the firm defaults and if the bubble bursts. The probability of both events is independent of the individual contract. The household will therefore use its bargaining power and set $\beta = Y\,l$. In a general bubble, the argument is more involved. Assume that there is a bubble in which ϕ is distributed such that some probability mass lies in a region just below the negotiated repayment β. In that case, knowing this distribution, households would prefer to choose a contract with a slightly lower β, such that they always obtain less from traders but economize on monitoring costs. The traders' participation would then not bind. Starting from such a distribution of ϕ, however, one can construct another distribution that also leads to market clearing but is strictly preferred by households. Specifically, if the probability mass from slightly below the default level is moved to the default level and some probability mass in the upper part of the distribution is moved to a lower level as a compensation, the asset market still clears, i. e., $\mathrm{E}[h(\phi)] = Y - \beta$. This distribution, together with the old β, is preferred by households as it increases the probability to obtain the original β. We additionally know from the above discussion that concentrating probability mass to the center of the distribution leads to a preferred distribution, seen from the perspective of the households. For the same reason, this distribution is itself dominated by the trinomial distribution. Summing up, even if bubble paths exist in which it is optimal for households not to push traders to their participation constraint, these bubbles are dominated by trinomial bubbles in which the participation constraint is binding. Hence, if households do not participate in trinominal bubbles, they do not participate in any other form of bubbles. Concentrating on the case of $\beta = Y\,l$ comes therefore without loss of generality. ■

Proof of Proposition 2. Consider condition (24). The effect of changes in λ is straightforward. When evaluating the effects of changes in the other parameters, however, we have to consider their effect on ϕ via the market clearing condition (20), $0 = q\,\phi^{-\gamma-1}(\phi-\beta)-(Y-\beta)$. Implicitly differentiating this equality shows that ϕ depends positively on Y and γ as well as negatively on β and q. Recall from the proof of Theorem 2 that the right-hand side of (20) depends positively on ϕ in the relevant region. Furthermore, the right-hand side of condition (24) depends negatively on ϕ and γ as well as positively on q, which completes the proof.[28] ■

Proof of Proposition 3. We start by looking at traders and households separately. Consider the initial generation at date $t = 0$. Traders invest a total amount of N dollars, including their own endowment. A fraction of p_t/N of these resources is invested in the risky asset in each period. Expected consumption of all N traders active at date 0 is therefore

$$\begin{aligned} E_0 C_0^T &= E_0 \left[N \frac{\overline{p}}{N} \left(\frac{p_0 + d}{\overline{p}} - \beta \right) + N \left(1 - \frac{\overline{p}}{N} \right) (Y - \beta) \right] \\ &= p_0 + d - \overline{p}\beta + (E_0 N - \overline{p})(Y - \beta). \end{aligned}$$

[28]When calculating the effect of γ, it is helpful to insert the transformed equality $\phi^{-\gamma-1} = q^{-1}(Y-\beta)/(\phi-\beta)$ into (24) to calculate the direct effect of γ and the indirect effect via ϕ on the resulting inequality. Observe that $\phi > Y$ because of the market-clearing condition.

Subtracting consumption in the steady state, we obtain traders' welfare difference between a bubbly situation and the steady state as

$$\hat{C}_0^T = p_0 - \overline{p} > 0.$$

Households, on the other hand, are not affected in the first period of the bubble. They get the contracted amount β, unless the firm underlying the risky asset defaults. As the probability for this event is independent of the existence of a bubble, the welfare difference $\hat{C}_0^H$ between a bubble and the steady state is zero. This changes in the following periods. Expected consumption of all N traders active at date 1 is

$$E_0 C_1^T = E_0 \left[N \frac{p_0}{N} Q_0 \left(\frac{p_1 + d}{p_0} - \beta \right) + N \left(1 - \frac{p_0}{N} \right) (Y - \beta) + N \frac{p_0}{N} (q - Q_0) \max \left(\frac{\overline{p} + d}{p_0} - \beta ; 0 \right) \right], \tag{36}$$

where p_1 is the expected price of the risky asset conditional on a continuation of the bubble and Q_0 the corresponding probability. The arbitrage condition for traders at date 0 can be stated as

$$Y - \beta = Q_0 \left(\frac{p_1 + d}{p_0} - \beta \right) + (q - Q_0) \max \left(\frac{\overline{p} + d}{p_0} - \beta ; 0 \right). \tag{37}$$

Combining the last two equations yields

$$E_0 C_1^T = E_0 N (Y - \beta).$$

Repeating the same steps for the steady state, this time inserting condition (5), shows that expected consumption is $E_0 N (Y - \beta)$ as well. The welfare difference for traders $\hat{C}_1^T$ is hence 0. Intuitively, traders are indifferent between investing in the safe asset or the risky asset in both the steady state and an ongoing bubble. The expected consumption difference is hence nil. This holds in the second period of the bubble and in all following periods.

Households, however, are hurt by limited liability of the traders. Expected consumption of a single household active at date 1 (one period after the bubble has taken off) is

$$E_0 \left[\frac{p_0}{N} \left(Q_0 \beta + (q - Q_0) \min \left(\frac{\overline{p} + d}{p_0} - c ; \beta - c \right) - (1 - q) c \right) + \left(1 - \frac{p_0}{N} \right) \beta \right],$$

where p_0/N denotes the probability that the commissioned trader invests in the risky asset. The second term in the square brackets denotes the household's payoff if the trader has invested in a bubble that then bursts. In this case, and if the underlying firm goes bankrupt, the household pays the monitoring costs c. Using the above equation to obtain expected consumption of all households and reformulating yields

$$E_0 C_1^H = p_0 Q_0 \beta + (q - Q_0) p_0 \left[\min \left(\frac{\overline{p} + d}{p_0} - \beta ; 0 \right) + \beta \right] + (E_0 N - p_0) \beta - p_0 (1 - Q_0) c. \tag{38}$$

In steady state, expected consumption of the household sector is $q\overline{p}\beta + E_0 (N - \overline{p}) \beta - \overline{p}(1-q)c$, such that the expected difference results as

$$E_0 \hat{C}_1^H = \beta [\overline{p}(1 - q) - p_0 (1 - Q_0)] + (q - Q_0) p_0 \left[\min \left(\frac{\overline{p} + d}{p_0} - \beta ; 0 \right) + \beta \right] - [p_0 (1 - Q_0) - (1 - q) \overline{p}] \, c. \tag{39}$$

Note that the term $-\beta p_0(1-Q_0)<0$ corresponds to the expected loss for the households due to the higher risk in a bubble. This becomes clear if we rewrite this term as $-\beta\,(1-Q_0)Np_0/N$, as $\beta=\beta$ is the amount claimed by one household, N the number of households, p_0/N the probability that traders invest household money in the bubble, and $1-Q_0$ the probability of a burst. This term therefore reduces households' expected utility in a bubble, compared to the steady state. As a mirror image, this expression equals the expected payoff gain due to limited liability for traders (with probability $1-Q_0$, they do not have to repay the contractual amount β), which demonstrates the link to the aggregate welfare of a generation. Given that limited liability imposes an externality on households not taken into account by traders, traders are induced to invest too many resources in the bubble.

Temporarily assuming that the household still obtains β if the bubble bursts, equation (39) becomes

$$\begin{aligned} E_0\hat{C}_1^H &= \beta[\bar{p}(1-q) - p_0(1-Q_0)] + (q-Q_0)p_0\beta - [p_0(1-Q_0) - (1-q)\bar{p}]\,c \\ &= \beta(1-q)(\bar{p}-p_0) - [p_0(1-Q_0) - (1-q)\bar{p}]\,c < 0. \end{aligned}$$

As more resources are channeled to the risky asset in a bubble, the risk that households are affected by a bankruptcy of the issuing firm increases. The term $\beta(1-q)$ represents this expected loss due to bankruptcy if funds are invested in the risky asset, net of monitoring costs. Taking into account the loss in case of a burst (that is, the fact that $\min[(\bar{p}+d)/p_0-\beta;0)]+\beta$ is weakly smaller than β) adds an additional negative component to expected household welfare.

Households of the next generation (and analogously of all following generations as long as the bubble continues) are therefore worse off compared to a situation without a bubble. The same applies to the aggregate of households and traders, since the traders from $t=1$ on experience the same welfare with and without a bubble. We hence obtain the result that only the initial generation of traders benefits from a bubble emergence. Households of the initial generation are not affected, while agents of future generations are (weakly) negatively affected in expectations by the existence of a bubble. ■

Proof of Proposition 4. Defining $Y'=(1+\tau')/(1+\tau)$ immediately shows that Proposition 1 applies. Concerning the participation constraint of households, we find that it might start to bind if taxes are levied on storage. The constraint in steady state is

$$\int_{\bar{p}(1+\tau')}^{\infty}\left[\frac{\bar{p}(1+\tau')}{N}(q\,\beta-(1-q)\,c)+\left(1-\frac{\bar{p}(1+\tau')}{N}\right)\beta\right]f(N)dN \geq l\,\lambda\,\frac{Y}{1+\tau}.$$

Households take the transfer of tax receipts from the government as given; it does hence not appear in the participation constraint. Evaluating the above integral shows that the participation constraint is independent of τ', as the probability that the trader invests into the risky asset is unaffected by this tax. Intuitively, while the amount money that flows into the asset (including taxes) is larger, the minimum $N_0=\bar{p}(1+\tau')$ increases proportionally. This implies that if the households participate in case no tax is implemented, they will continue to do so for a rising tax rate. Placing a tax on the safe asset, on the other hand, reduces the payoff from the household's alternative investment possibility. At the same time, it reduces $\beta = Yl/(1+\tau)$. It is straightforward to verify that, for $c > 0$, the participation constraint of households might be violated for high enough values of τ. In a bubble, the expected payoff from lending to traders is lower than in steady state. As in the steady state, the household obtains β in case the trader does not default, but the risk of a default increases. Hence, if a tax on the safe asset violates the participation constraint for households in steady state, it will also do so in a bubble. Taken together, we find that a tax on the safe asset can create or eliminate the possibility of bubbles. The latter, however, works only via destroying the intermediation market. ■

Proof of Proposition 5. Regarding welfare consequences, we know from the proof of Proposition 4 that increasing τ' can render bubbles impossible and vice versa for τ. We have to additionally check, however, if a higher τ' or τ will lead to worse outcomes outside a bubble, i. e., in steady state. Total expected consumption of future generations is

$$E_0 C^n = E_0 \frac{Y+\tau}{1+\tau}[N - \bar{p}^n(1+\tau')] + q(\bar{p}^n + d) + \tau'\bar{p}^n - \bar{p}^n(1-q)c,$$

where C^n is consumption with the tax, $\bar{p}^n$ denotes the steady-state price that obtains with taxes, $N - \bar{p}^n(1+\tau')$ is the expected amount invested in the safe asset, $q\bar{p}^n$ is the expected reselling value, and $\tau'\bar{p}^n + \tau[N - \bar{p}^n(1+\tau')]/(1+\tau)$ is the tax return to the household. The steady-state price with taxes, observing $\beta = Y\,l$, is

$$\bar{p}^n = \frac{dq}{\frac{1+\tau'}{1+\tau}Y[1 - l(1-q)] - q}.$$

Its derivative w. r. t. τ' is negative and positive w. r. t. τ. The derivative of E_0C^n w. r. t. τ', setting $\tau = 0$ for simplicity, is

$$\frac{\partial E_0 C^n}{\partial \tau'} = \bar{p}^n(1-Y) + [\tau'(1-Y) + q - Y - (1-q)c]\frac{\partial \bar{p}^n}{\partial \tau'}.$$

This expression is positive, which can be shown by inserting the above value for $\bar{p}^n$, and using

$$\frac{(1-q)(1+c)}{1-Y}Y[1 - l(1-q)] < q.$$

Consumption in steady state hence increases with higher transaction taxes on the risky asset. Knowing from Proposition 3 that welfare in steady state is higher if compared to a bubbly situation with the same parameter values (except for the initial generation), we can conclude that a policy measure that can prevent bubbles *and* increases steady-state welfare is unambiguously welfare enhancing for future generations, as long as the tax is high enough to prevent bubbles. The derivative of C^n w. r. t. τ, on the other hand, is

$$\frac{\partial C^n}{\partial \tau} = E_0[N - \bar{p}^n(1+\tau')]\left(1 - \frac{Y+\tau}{1+\tau}\right)/(1+\tau) + \frac{\partial \bar{p}^n}{\partial \tau}\left(q - \frac{Y+\tau}{1+\tau} - \tau'\frac{Y-1}{1+\tau} - (1-q)c\right) < 0,$$

demonstrating that a tax on the safe asset reduces steady-state welfare. Lastly, for $\tau' = \tau$, we obtain

$$\frac{\partial C^n}{\partial \tau} = \frac{E_0 N}{1+\tau}\left(1 - \frac{Y+\tau}{1+\tau}\right) < 0,$$

which shows that a tax on both assets reduces steady-state welfare as well. ■

Proof of Proposition 6. Repeating the steps in the proof for Theorem 1, assuming that each trader can borrow the maximum amount of l, yields the following conditions for the possibility of bubbles

$$l > \frac{\gamma}{Y(\gamma+1)}$$

$$l^\gamma - l^{\gamma+1} \leq q\frac{\gamma^\gamma}{[Y(\gamma+1)]^{\gamma+1}},$$

where we take the endogeneity of β into account. The participation constraint of traders with capital requirements changes to $eY \geq (e+l)Y - \beta$, as traders can obtain eY by investing their own funds only, or $(e+l)Y$ by additionally borrowing from households. Households will therefore set again $\beta = Yl$. Reducing l by introducing capital requirements will hence destroy the possibility of bubbles, as at least one of the two above conditions will be violated for a certain value of $l > 0$. As we only want to show that capital requirements can prevent bubbles, we don't need to verify households' participation constraint. ■

Proof of Proposition 7. We compare two situations, the old steady state without capital requirements and a steady state with capital requirements. Expected aggregate period consumption in a given period with the capital requirement in place is

$$E_0 C^n = \lambda Y\, E_0 N + (1-\lambda)Y(e+l)E_0 N + \bar{p}^n q + q\, d - \bar{p}^n\, Y - \bar{p}^n (1-q)\, c, \qquad (40)$$

where $\lambda Y E_0 N + (1-\lambda)Y(e+l)E_0 N - \bar{p}^n Y = E_0 Y[N(e+l) - \bar{p}^n] + \lambda Y E_0 N(1-e-l)$ is the expected amount invested by the traders into the safe asset plus the investment of the household into the inferior investment technology. $\bar{p}^n$ is the steady-state price with the policy in place, $\bar{p}^o$ the one without. In the following, all $\bar{p}$ denote $\bar{p}^n$, except where explicitly mentioned. $\bar{p}q$ is the expected revenue from selling the asset, and qd the expected dividend. The derivative of the consumption difference between the new and the old steady state is

$$\frac{\partial C^n - C^o}{\partial l} = (1-\lambda)Y\, E_0 N - (Y - q + (1-q)\, c)\partial \bar{p}^n / \partial l.$$

The steady-state price (6) becomes

$$\bar{p} = \frac{dq}{Y - q - Y(1-q)l/(e+l)},$$

where the outcome $\beta = Y\, l$ of negotiations is already inserted. The derivative w. r. t. l is

$$\frac{\partial \bar{p}}{\partial l} = \frac{Y(1-q)\bar{p}}{Y - q - Y(1-q)l/(e+l)} \frac{e}{(e+l)^2} > 0.$$

Combining these equations yields

$$\frac{\partial C^n - C^o}{\partial l} = \frac{Y(1-q)\bar{p}(q - Y - (1-q)c)}{Y - q - Y(1-q)l/(e+l)} \frac{e}{(e+l)^2} + (1-\lambda)YE_0N,$$

which is negative (higher capital requirements lead to welfare improvements in steady state) if

$$\frac{(1-q)\bar{p}(Y-q)}{Y - q - Y(1-q)l/(e+l)} \frac{e}{(e+l)^2} > (1-\lambda)E_0N. \tag{41}$$

This condition holds for certain parameter constellations; only if $q = 1$ there is no region in which this inequality is fulfilled (as $\lambda \leq 1$). In this case, the steady-state price does not depend on l, since there is no limited liability due to the lack of risk. Lowering l is then unambiguously bad.

From the above it is not clear if welfare increases or decreases relative to the old steady state with the introduction of capital requirements. On the one hand, they can destroy the possibility of bubbles. On the other hand they can decrease welfare because the household has to use the inferior investment technology. The lower steady-state price (once the bubble has collapsed, if it ever took off) is again beneficial for future generations because less is invested into the risky asset. We can make some statements about inequality (41) if we use the participation constraint of households. Expected payoff in steady state for a household is

$$\int_{N_0}^{\infty} \left[\frac{\bar{p}}{N(e+l)} (q\,Y\,l - (1-q)\,c) + \left(1 - \frac{\bar{p}}{N(e+l)}\right) Yl \right] f(N)dN + (1 - e - l)\lambda Y,$$

where the probability that the trader invests into the asset $E_0\bar{p}/(N(e{+}l))$ is adjusted for the fact that the traders can now invest less money and N_0 is the highest level of N that exists for sure. In this context, $N_0 = \bar{p}/(e{+}l)$, as the total amount of funds that traders control (that is the maximum amount that all traders could have invested into the asset) is $N(e{+}l)$ and we know that $\bar{p}$ was invested into the asset. The above expression has to be larger than the alternative investment return of the household without lending to traders, which is $l\lambda Y$. We therefore get

$$\int_{N_0}^{\infty} \frac{\bar{p}}{N(e+l)}(q-1)(Y\,l + c)f(N)dN + \int_{N_0}^{\infty} Ylf(N)dN \geq l\,\lambda\,Y$$

$$\Longrightarrow \frac{\bar{p}}{e+l}(1-q)\left(1 + \frac{c}{Y\,l}\right) \int_{N_0}^{\infty} \frac{1}{N} f(N)dN \leq 1 - \lambda. \tag{42}$$

Taking the derivative of the left-hand side of this equation w. r. t. l, we obtain

$$\left[\frac{\partial \bar{p}/(e+l)}{\partial l} \left(1 + \frac{c}{Yl}\right) - \frac{\bar{p}}{e+l} \frac{c}{Yl^2} \right] (1-q) \int_{N_0}^{\infty} \frac{1}{N} f(N)dN < 0.$$

Decreasing l from a situation in which the household participates leads therefore at some point to a violation of the household participation constraint. A reduction in l results in an increase in the share of traders' funds flowing to the risky asset – the derivative of $\bar{p}/(e{+}l)$ w. r. t. l is negative. The household hence needs to invest more in the inferior investment technology (involuntarily, as shown by revealed preferences in the initial situation) *and* traders' behavior becomes riskier.

Finally, we compare the condition required for steady-state welfare to depend negatively on l, equation (41), with the participation constraint of households, equation (42). If both are fulfilled simultaneously, we can insert the latter into the former,

$$\frac{Y-q+(1-q)c}{Y-q-Y(1-q)l/(e+l)}\frac{e}{e+l} > \left(1+\frac{c}{Yl}\right)\int_{N_0}^{\infty}\frac{1}{N}f(N)dN\int_{N_0}^{\infty}Nf(N)dN. \tag{43}$$

Since

$$\int_{N_0}^{\infty}\frac{1}{N}f(N)dN\int_{N_0}^{\infty}Nf(N)>1 \quad \text{and} \quad \frac{(1-q)ce}{(Y-q)(e+l)-Y(1-q)l}<\frac{c}{Yl}$$
$$\text{and} \quad ql(Y-1)>0 \iff \frac{Y-q}{Y-q-Y(1-q)l/(e+l)}\frac{e}{e+l}<1,$$

we conclude that inequality (43) does not hold and hence conditions (41) and (42) cannot be fulfilled simultaneously. That is, if households participate, reducing l also reduces welfare. If households do not participate anymore, reducing the maximum l has no further effects. We thus get a negative effect of capital requirements on welfare in steady state. ■

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