Auerbach, Lotka, Zipf – pioneers of power-law city-size distributions

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Power-law city-size distributions are a statistical regularity researched in many countries and urban systems. In this history of science treatise we reconsider Felix Auerbach's paper published in 1913. We review his analysis and find (i) that a constant absolute concentration, as introduced by him, is equivalent to a power-law distribution with exponent ≈ 1, (ii) that Auerbach describes this equivalence, and (iii) that Auerbach also pioneered the empirical analysis of city-size distributions across countries, regions, and time periods. We further investigate his legacy as reflected in citations and find that important follow-up work, e.g. A.J. Lotka 1925 and Zipf 1949, does give proper reference to his discovery – but other does not. For example, only approximately 20 % of city-related works citing Zipf 1949 also cite Auerbach 1913. To our best knowledge, A.J. Lotka 1925 was the first to describe the power-law rank-size rule as it is analyzed today. M. Saibante 1928, building on Auerbach and Lotka, investigated the power-law rank-size rule across countries, regions, and time periods. G.K. Zipf's achievement was to embed these findings in his monumental 1949 book. We suggest that the use of "Auerbach-Lotka-Zipf law" (or "ALZ-law") is more appropriate than "Zipf's law for cities", which also avoids confusion with Zipf's law for word frequency. We end the treatise with biographical notes on Auerbach.

Broad city-size distributions are a well investigated phenomenon and are often studied by means of rank-size representations. Their discovery is mostly attributed to G.K. Zipf and his monumental book "Human behavior and the principle of least effort: An introduction to human ecology" [1]. While their origin and some aspects, such as log-normal vs. power-law, upper or lower cut-offs, specific value of the exponent, are contested [2], there is a general agreement about the broad nature of city-size distributions [3, e.g.]. In contrast to e.g. the normal distribution, cities or settlements occur in a range of sizes and it is characteristic that there are many more small than large ones.

Here we trace back the pioneering works that deal with this phenomenon and find that for city sizes it was first discovered by F. Auerbach in 1913 [4]. The rank-size representation was introduced by A.J. Lotka in 1925 [5] and the topic was popularized by G.K. Zipf [1, 6, 7]. In the following we want to review Auerbach's original work, discuss some bibliometric analysis, and speculate about reasons why the Auerbach' work receives so much less attention than Zipf's. We also discuss the contribution of other contemporaries, in particular Saibante, Christaller, and Gibrat. We end with biographic notes on Auerbach who is the least known among these names [8].

In his paper "Das Gesetz der Bevölkerungskonzentration" (The law of population concentration) [4] Auerbach defines a quantity which he calls "absolute concentration" (A.K.) given by the product of rank r and city size s. The size of cities is given by their population and the rank is obtained by ranking cities according to their size, where the largest city has rank 1. Plotting AK vs. r for German cities in 1910, he finds that AK reaches, after some deviations at lower ranks, a plateau of approximately constant value. Today the rank-size relation

$$s \sim r^{-\alpha} \tag{1}$$

is studied and reflects a power-law city-size distribution [9] analogous to the Pareto distribution for income [10]. Combining Auerbach's AK and Eq. (1) leads to $\mathrm{AK} = rs \sim r^{1-\alpha}$, which is constant for $\alpha \simeq 1$. Accordingly, Auerbach's finding of an approximately constant AK across German cities is equivalent to what is known today as "Zipf's law for cities". As becomes clear from his final paragraph, Auerbach was aware that constant $r \times s$ characterizes the size distribution and he already called it a law – according to which the count of cities is inversely proportional to their minimum size [11].

In order to make comparisons across countries and time periods, Auerbach suggests to divide each country's AK by its population and thereby to obtain what he calls specific concentration (Sp.K.) as $\mathrm{SpK} = \frac{AK}{S}$, where S is the total population of the country. He then uses SpK to investigate city size distribution across countries, see Fig. 1(a), regions, and time periods.

To our best knowledge, A.J. Lotka 1925 [5] was first to show the rank-size plot, see Fig. 1(b), and to mention the power-law relation as it is analyzed today [12, e.g.]. Lotka's empirical rank-size representation is a simple and elegant method given the means that were available at

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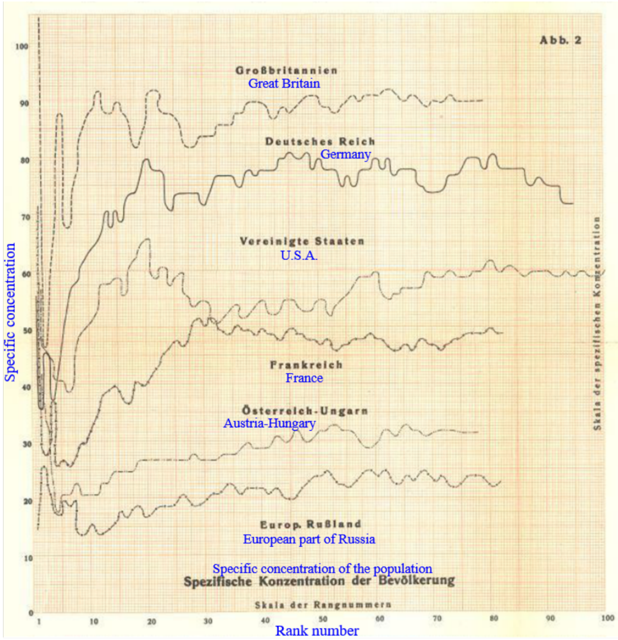

(a) Auerbach 1913 [4, 11]

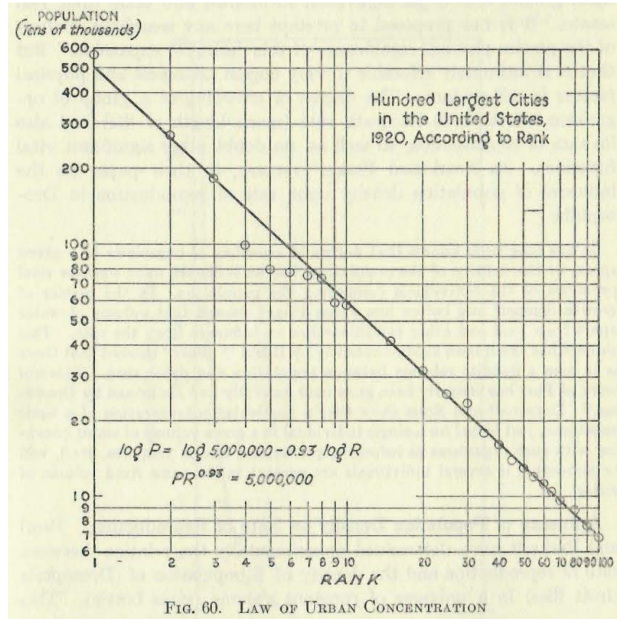

FIG. 60. LAW OF URBAN CONCENTRATION

(b) Lotka 1925 [5]

Figure 1: Original graphical representations. (a) Auerbach plots the specific concentration (city population times rank divided by country population) as a function of the rank (for various countries). A constant behavior corresponds to Eq. (1) with $\alpha \simeq 1$. (b) Lotka simply plots the city population vs. rank (for USA cities). The straight line in log-log representation corresponds to Eq. (1). Lotka measures $\alpha \simeq 0.93$.

the time. It also draws the attention to estimates of the exponent instead of Auerbach's AK and SpK. However, in his book "Elements of Physical Biology" [5] Lotka treats a range of ecological problems and the "Law of Urban Population" seems a bit displaced between "The Intensity Law in Organic and Economic Systems" and "The Biological Background of Population Pressure".

A little known but rather early and extensive work is the paper by M. Saibante published in 1928 [13]. Saibante built on Lotka's formulation and investigated how well it fits city-size distributions across a range of countries, regions, and time periods. He focused on the exponent, see Fig. 2, and thereby laid the foundation of modern empirical investigations. Saibante drew the connection to the Pareto distribution and additionally calculated the Gini coefficient. We speculate that as an Italian he had better access to their work.

Zipf wrote three publications on city-size distributions. In 1941 his book "National unity and disunity" came out and almost exclusively treats cities and rank-size plots but only includes a small number of equation and references. In reaction to a book review by Thorndike [14], Lotka clarifies that the discovery was first indicated by Auerbach [15]. The second publication is Zipf's 1942 paper [7] in which he admits that the discovery was made by Auerbach and he regrets not knowing the reference before publishing [6]. In this treatise we focus on his third publication, his well-known 1949 book [1], because it seems more elaborated and most cited today.

Valori di α nelle singole Nazioni.

NAZIONI	Anno di rilevazione	α
Confederazione Australiana	1921	0,82
Finlandia	1923	0,94
Canadà	1920	0,98
Cile	1920	1,01
Stati Uniti	1921	1,03
Inghilterra	1911	1,04
Germania	1925	1,11
Danimarca	1925	1,13
Svezia	1920	1,17
Jugoslavia	1921	1,17
Francia	1921	1,24
Italia	1921	1,29
Polonia	1921	1,36
Olanda	1921	1,39
Spagna	1921	1,45
Belgio	1921	1,45
India Inglese	1921	1,68

Figure 2: Original table from Saibante 1928. For various countries the value of α is listed. Please note, Saibante swapped the axes compared to Lotka so that the α reported here is the inverse of the one in Eq. (1). Source: [13, p.58].

Zipf not only presents empirical analysis but also some sort of explanation for the power-law rank-size relation – he frames it as a manifestation of an optimization ("least effort"). Specifically, he introduces a model of economic geography where the urban system is organized in such a way that costs of production and consumption are minimized in terms of labor and transport [1, Ch.9, Sec.I]. The model leads to a set of predictions, e.g. regarding the diversity (of establishments) as a function of city size. He argues that if transport costs are high, then people migrate to production centers; if they are low, then goods are brought to the consumers. Zipf calls these two effects "Force of Diversification" and "Force of Unification", respectively. In the extreme case, they lead to a large number of small settlements or a small number of large cities, respectively. The power-law rank-size relation is then introduced as a straight line in double-logarithmic representation, connecting these two extremes as supporting points, which leads to $rs^{\beta} = \text{const}$ [1, Eq.(9-12)], corresponding to Eq. (1) with $\beta = 1/\alpha$ [16, e.g.]. Moreover, Zipf attributes the exponent β to the ratio of the magnitude of both forces. Empirically, Zipf estimates $\alpha \approx 1$ (implying $\alpha \simeq \beta$) which he interprets as balance between diversification and unification [1, Ch.9, Sec.III].

Table I provides an overview of early works and the referencing among them. It represents a subjective set of works related to cities and city-size distributions. Auerbach [4] is cited in the later publications but the economics works Gibrat [17], Singer [18], and Lösch [19] represent an exception. Lotka [5] is only cited by the latest publications Stewart [20] and Zipf [1] (the paper by Saibante [13] got very little attention). Almost all works somehow refer to Pareto [10] but to our best knowledge, Saibante [13] was the first to recognize the connection between Auerbach's and Pareto's works. Singer [18] was the first to spell out the "complete analogy" in an English publication. There seems to be a parallel evolution of two groups, i.e. works cited by Zipf [1] and those cited by Lösch [19].

Gibrat [17] is off in Tab. I as he describes the log-normal distribution. Therefore, he analyzed the (complementary) cumulative distribution function (i.e. counts of cities that are more populous than certain threshold values x) and from the respective probabilities he obtained corresponding values z from the tabulated probit function. He then found a logarithmic relation $z = a \log x + b$, where a and b are parameters. A logarithmic argument in the normal distribution corresponds to a log-normal distribution. As z vs. $\log x$ is a linear function, it has been misquoted by Singer [18] ("Gibrat's formula is more general") and Zipf [1] ("Gibrat reported the rectilinearity") but Lösch [19] points towards "differences between Gibiat's and Pareto's methods".

To asses the impact that Auerbach's work had and still has, we analyze the number of citations that Auerbach 1913 [4] and Zipf 1949 [1] receive across the years. Therefore, we use the Publish or Perish software [28] to retrieve the bibliographic information (from google-scholar; QueryDate: 2021-07-15) of those works citing Auerbach (A13) or Zipf (Z49). Since the Zipf-book is also about word frequency and other topics not immediately related to cities, it also receives unrelated citations. Accordingly, we only consider citations where the following keywords appear in the title or in the beginning of the abstract: city, cities, town, towns, village, villages, settlement, settlements, metropolis, metropolises, urban, urbanization, urbanisation, population, populations. To be consistent we apply the same filtering to both works and the number of counts reduces from 868 to 334 for A13 and from 15,339 to 1,368 for Z49 (227 publications cite both). In Fig. 3(a) we plot the number of citations per year for either and both. In Fig. 3(b) the share of Z49-citations that also cite A13 is plotted. We can make three observations: (i) Works citing Auerbach 1913 [4] to a large extent also cite Zipf 1949 [1] – but less so vice versa. (ii) Typically 20 % of the works citing Zipf 1949 [1] also cite Auerbach 1913 [4]. (iii) There are two waves, the first peaking around 1980 (possibly related to the Simon and Mandelbrot dispute in the 1960s [29]) and the second one around 2015. We can conclude that the discovery is mostly attributed to Zipf and only a relatively small share acknowledges Auerbach. (Cottineau [30] reviewed 66 sources and found that most works not citing Zipf were published before 1990.)

This leads to the question why Auerbach's paper [4] is so much less recognized although it was published more than 30 years before Zipf's book [1]. First, instead of Auerbach focusing on the regularity itself, he uses it as a prerequisite for his measures of absolute and specific concentration which are only meaningful if the regularity holds true. Moreover, his measures of absolute and specific concentration were found to have drawbacks [31] and did not catch on. Second, Auerbach's paper is written in German [32] and Zipf's work is written in English which was/is advantageous when English took the role of (global) science language. Third, Zipf's works are more extensive compared to Auerbach's 2-page paper (plus Figures). In addition, Zipf also finds a power-law in the word occurrence and discusses city-size distributions as a consequence of "the principle of least effort". A fourth reason could be that early economics work in the context did not cite Auerbach [4] and this missing reference could have been reflected in follow-up work in the rather big economics field. Lösch even attributed the discovery to Singer [18], see Tab. I.

Interestingly, Christaller 1933 [22], the creator of *Central Places Theory*, gives reference to Auerbach 1913 [4] but in an ambiguous way. On the one hand, he writes "≫Auerbach's law≪ ... is not much more than playing with numbers." (note 19, p.82). On the other hand, he also states that explaining Auerbach's finding is one of his major goals. "We come now to the chief problems of our investigations. ... generally and less exactly speaking, besides a great number of the smaller and smallest towns and market places, there are only a small number of greater towns; and the greater a town is, the smaller

Author	Year	Ref.	Trans.	Type	[10]	[4]	[5]	[21]	[13]	[17]	[22]	[18]	[19]	[20]	[1]	Comment
Pareto	1896	[10]		book	○											French
Auerbach	1913	[4]	[11]	paper		○										German, no reference at all
Lotka	1925	[5]	-	book	·	×	○									
Goodrich	1926	[21]	-	paper				○								no references at all
Saibante	1928	[13]		paper	×	×	×		○							Italian, first connection to Pareto 1896 [10]
Gibrat	1931	[17]		book	·				a	○						French, finds log-normal distribution
Christaller	1933	[22]	[23]	book		×					○					German
Singer	1936	[18]	-	paper	×			b		×		○				
Lösch	1940	[19]	[24]	book	×					×	×	×	○		c	German, mentions “Singer’s discovery”
Stewart	1947	[20]	-	paper	·	×	×				·			○	c	
Zipf	1949	[1]	-	book	·	×	×	×		×	·			·	○	mentions “Pareto school”

Table I: Citation matrix. ×, ‘proper’ referencing; ·, either mentioning or citing in a different context; ○, diagonal elements; [a], Saibante 1926 [25]. [b], Goodrich 1928 [26]. [c], Zipf 1941 [6]. Lösch’s book first appeared in 1940 but we consider the translation which is based on the 1944 2nd edition. We would like to note that we compiled this table with the best of our knowledge but cannot exclude that we miss a citation. “Ref.” stands for reference and “Trans.” for translation to English.

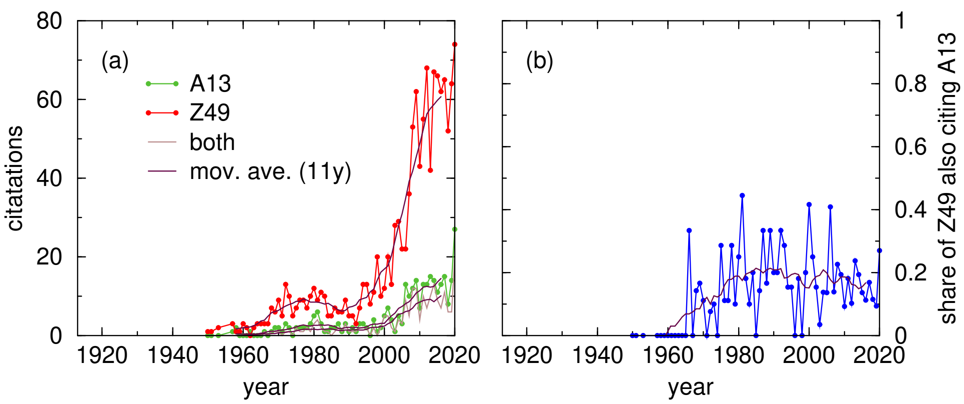

Figure 3: Citations to Auerbach 1913 [4] (A13) and Zipf 1949 [1] (Z49). (a) Citations per year individually and both. Please be aware of the approximately exponentially increasing global publishing activity [27]. (b) Share of works citing Auerbach that also cite Zipf. In both panels, the dark solid line represents an 11y moving average. Almost all works citing Auerbach also cite Zipf – but not so vice versa.

is the number falling in its respective category. This fact has already led to the statement of a most incredible law. ... Is there a possibility of truly explaining this fact?” At the same time he wants to search for such a law “Are there rules governing the relations of the frequency of a given size-class of city to the frequencies of other size-classes?” (“8. The System of Central Places”, p.58ff). Not surprisingly, it was confirmed that a special case of Christaller’s theory is consistent with the rank-size relation [33]. It seems, Christaller did not recognize Auerbach’s achievement but nevertheless appreciated the importance of his discovery.

While Christaller made a career under the Nazi-regime [34], Lösch, who published several years after Christaller, avoided compromises with the regime [24, in Memoriam] and died at young age of scarlet fever shortly after the end of World War II. Although Auerbach (born 1856) accomplished his habilitation at an early age of 22, only in 1889 did he become an extraordinarius professor for theoretical physics at University of Jena. As detailed in [35], his assignment was obstructed because of antisemitism in the department [36, 37, see also]. Only in 1923, shortly before his retirement in 1927, was he promoted to an ordinarius professor [35]. In 1933, after his second stroke and the Nazis rise to power, Auerbach and his wife Anna (née Silbergleit) committed suicide [37, e.g.].

Having published more than 20 books, Auerbach was a versatile physicist and scholar (further biographical details can be found elsewhere [36–40, e.g.]). His work on the measurement of hardness [41] led to an Auerbach’s law in physics [35, 42, e.g.]. Auerbach’s lectures at University of Jena were attended by the philosopher Rudolf Carnap [43] and his diaries suggest interdisciplinary syllabi. As Auerbach published on core physics topics, including relativity theory, *and* on musical art and fine art from the point of view of the natural scientist, Müller

Figure 4: Portrait of Felix Auerbach by Edvard Munch, 1906, oil on canvas, 85.4 cm × 77.1 cm, Van Gogh Museum Amsterdam.

[44] argues that he served as mediator between Einstein and the Bauhaus [45, see also]. He served as a patron of the arts [37, 39, 44, 46]. A portrait of him by Edvard Munch [37, 46], see Fig. 4, is an example of his engagement in the contemporary art scene. Auerbach also commissioned the Bauhaus architect Walter Gropius (with Adolf Meyer) to build his house in Jena. It was finished in 1924 and turned out to be the first single-family house following the *New Building* principles [37].

To date, there is an extensive body of literature, including reviews [2, 47], meta studies [12, 48], international comparisons [49, 50], and possible explanations [33, 51, e.g.] of ALZ-law. Future work could inspect more closely possible conceptual predecessors, including Reynaud 1841 [52] and Kohl 1850 [53], as well as successors of the second wave, including [16, 33, 54–56]. One theoretical strand are models employing stochastic mechanisms (i.e. *proportionate effect* resp. *preferential attachment*) and evolving around Yule [57, 58], Gibrat ("Gibrat's law") [17, 59], and Simon [60]. An overview can be found in [29]. It is also worth mentioning that broad distributions are abundant in nature and society [61, e.g.], and a different perspective could be gained when approaching the phenomenon from other disciplines, such as linguistics or ecology.

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