Completely Regular Functions: a Comment on Monotone Comparative Statics

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Abstract

Milgrom and Shannon [1994] have shown the equivalence of the (strict) single crossing property and the (strict) Spence-Mirrlees condition for a class of completely regular C^2 functions. Edlin and Shannon [1998] have proven that the equivalence holds only in the weak cases, while with strict conditions only a one-sided implied relationship holds. This article shows the importance of complete regularity assumption.

Key Words: Spence-Mirrlees, single crossing, monotone comparative statics, strategic complementarity, super-modularity

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1 Completely regular or not completely regular?

The main contribution of this article is to test the validity of Theorem 1 below (about the equivalence of the Spence-Mirrlees condition and the single crossing property) when applied to the class of functions that are C^1 but not completely regular, which Milgrom and Shannon [1994] define as follows:

Definition 1 (Milgrom and Shannon (1994) definition). If a function f is continuously differentiable and the indifference sets of f are path-connected, then f is said to be *completely regular*.

As I will show below, the complete regularity assumption or, to be more precise, the assumption of path-connected indifference sets, is critical for the theorem to hold. The assumption of differentiability cannot be dropped, since the definition of the Spence-Mirrlees condition requires functions to be differentiable. The assumption of countinuous differentiability is not investigated here.

Before proceeding any further, some definitions and the main theorem, due to Milgrom and Shannon [1994] and Edlin and Shannon [1998], should be given:

Definition 2. Let $f : X \times T \longrightarrow \mathbb{R}$ be continuously differentiable, where $X \subset \mathbb{R}^2$. Then f is said to satisfy the *(strict) Spence-Mirrlees condition* if $f_x/|f_y|$ is (increasing) nondecreasing in t, and $f_y \neq 0$ and has the same sign for every (x, y, t).

Definition 3 (Milgrom and Shannon (1994) definition). Let X and T be partially ordered sets. A function $f : X \times T \longrightarrow \mathbb{R}$ is said to satisfy the *single crossing property* in $(x; t)$ if for all $x' > x^*$:

1. whenever $f(x', t^*) \geq f(x^*, t^*)$, then $f(x', t') \geq f(x^*, t')$ for all $t' > t^*$; and

2. whenever $f(x', t^*) > f(x^*, t^*)$, then $f(x', t') > f(x^*, t')$ for all $t' > t^*$.

The function is said to satisfy the *strict single crossing property* in $(x;t)$ if for all $x' > x^*$, whenever $f(x',t^*) \geq f(x^*,t^*)$, then $f(x',t') > f(x^*,t')$ for all $t' > t^*$.

Theorem 1 (Milgrom and Shannon (1994) Theorem 3, Edlin and Shannon (1998) Theorem 2.1). *Let $\mathbb{R}^2$ be given lexicographic order, with $(x,y) \geq (x',y')$ if either $x > x'$ or $x = x'$ and $y \geq y'$. Suppose that $U(x,y;t) : \mathbb{R}^2 \times \mathbb{R} \longrightarrow \mathbb{R}$ is completely regular with $U_y \neq 0$. Then $U(x,y;t)$ has the single crossing property in $(x,y;t)$ if and only if it satisfies the Spence-Mirrlees condition. Moreover, $U(x,y;t)$ has the strict single crossing property in $(x,y;t)$ if it satisfies the strict Spence-Mirrlees condition.*

Below, I will construct a function which has disconnected indifference sets, but otherwise satisfies all of the prerequisites of the theorem. As I will show, such a function fails to have the single crossing property even though the Spence-Mirrlees condition is satisfied everywhere.

The idea is to construct a function similar to $F(x,y,t)$ shown in Picture 1. For $F = 0$ and above, say, this function has disconnected level curves with asymptotes at $x = 0$ and $y = 0$. For this function to behave in accordance with the Spence-Mirrlees condition, the level curves of $F > 0$ have to be rotated a bit in a clockwise direction (with the change in t), but retain asymptotes as previously, and not cross $F = 0$. Suppose, further, there are points $(\tilde{x},\tilde{y})$ and $(\bar{x},\bar{y})$ with $\tilde{x} < 0 < \bar{x}$, and $\tilde{t} < \bar{t}$, such that $F(\tilde{x},\tilde{y},\tilde{t}) = F(\bar{x},\bar{y},\tilde{t}) = \tilde{F}$ and $F(\tilde{x},\tilde{y},\bar{t}) = F(\bar{x},\bar{y},\bar{t}) = \bar{F}$. Then there must be a point (x^*,y^*), as shown in Picture 1, such that $x^* > \tilde{x}$ and $F(x^*,y^*,\tilde{t}) > F(\bar{x},\bar{y},\tilde{t}) = F(\tilde{x},\tilde{y},\tilde{t}) = \tilde{F}$, but $F(x^*,y^*,\bar{t}) < F(\bar{x},\bar{y},\bar{t}) = F(\tilde{x},\tilde{y},\bar{t}) = \bar{F}$, thus violating the single crossing property. I shall now construct such a function F.

Consider the following example.

Let $(x,y;t) \in \mathbb{R}^2 \times \mathbb{R}$ and $\mathbb{R}^2$ be equipped with the lexicographic order. Let the functions u and g be defined as follows:

$$u(x,y) = yx^2 - e^{-y} \tag{1}$$

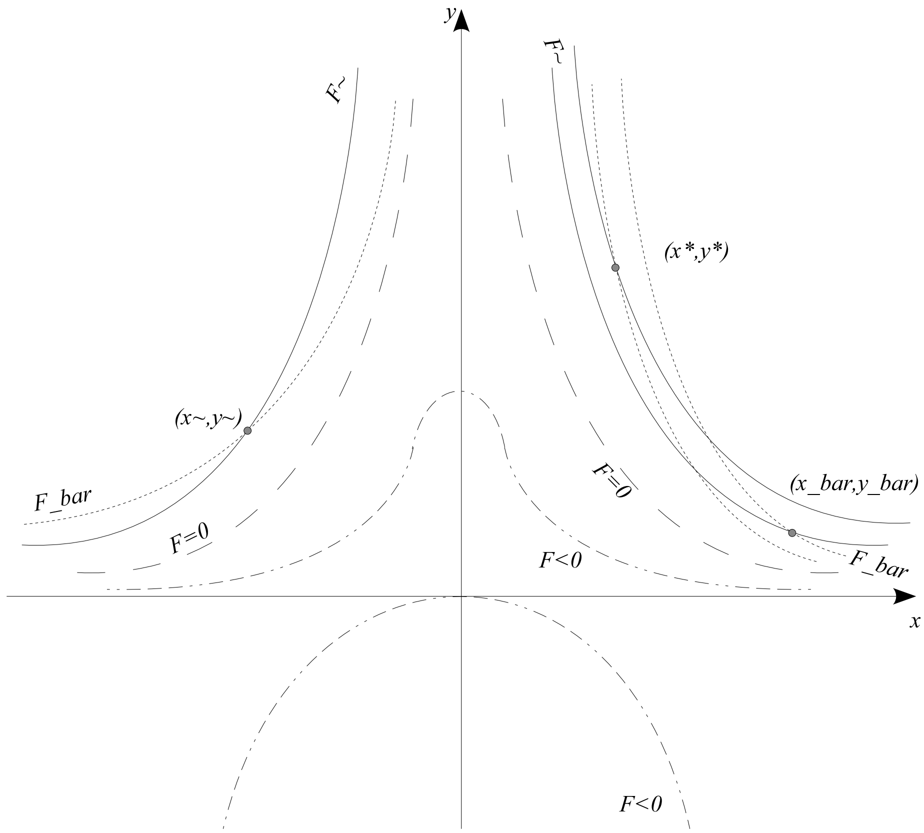

Figure 1: Function F

$$g(t) = \frac{\arctan(t)}{\pi} + 1.5 \tag{2}$$

Note that $g(t)$ is a strictly increasing function, bounded between 1 and 2, and $u(x, y)$ is C^∞ with $\frac{\partial}{\partial y} u(x, y) = x^2 + x^{-y} > 0$. Furthermore, $u(x, y)$ is not completely regular, for it has disconnected indifference sets for all $u(x, y) > 0$.

Further, define functions ψ and φ as follows:

$$\psi(u(x, y), g(t)) = \begin{cases} 1 & u(x, y) \leq 0 \\ 1 - e^{\frac{-g(t)}{u(x,y,)^2}} & u(x, y) > 0 \end{cases} \tag{3}$$

$$\varphi(u(x, y), g(t)) = \begin{cases} 1 & u(x, y) \leq 0 \\ 1 + e^{\frac{-g(t)}{u(x,y,)^2}} & u(x, y) > 0 \end{cases} \tag{4}$$

Here, both functions are C^∞ not only in their own arguments, but also in x, y and t, with $\psi(u(x, y), g(t))$ being strictly decreasing in u for $u > 0$ and bounded between 1 and 0, for all t, and $\varphi(u(x, y), g(t))$ strictly increasing for $u > 0$ and bounded between 1 and 2.

To show that ψ is C^∞ in x, y and t, consider the following construction.

Define $A : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \times (1, 2)$ as follows: $(x, y, z) \longmapsto (u(x, y), g(z))$. Since both u and g are C^∞, A is also C^∞.

Now, define $B : \mathbb{R} \times (1, 2) \longrightarrow \mathbb{R}$ by if $x > 0$, $(x, y) \longmapsto (1 - e^{-y/x^2})$, otherwise $(x, y) \longmapsto 1$. The function e^{-y/x^2} is C^∞, and so is $1 - e^{-y/x^2}$, for any y and for all $x > 0$. But when $x \rightarrow_{-} 0$ and for any y, $1 - e^{-y/x^2}$ tends to 1, while e^{-y/x^2} tends to 0, as does every partial derivative of e^{-y/x^2}. And, when $x \leq 0$, then the function is constant, with all derivatives equal to 0 everywhere. So, B is C^∞.

But then the composite function $B(A(x, y, t)) \equiv \psi(x, y, t)$ must also be C^∞ for all $(x, y, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

Similarly, φ is also C^∞.

Now, define function $U(x, y; t) : \mathbb{R}^2 \times \mathbb{R} \longrightarrow \mathbb{R}$ by:

$$U(x,y;t) = \begin{cases} yx^2 - (e^{-y})\psi(u(x,y),g(t)) & x < 0 \\ yx^2 - e^{-y} & x = 0 \\ yx^2 - (e^{-y})\varphi(u(x,y),g(t)) & x > 0 \end{cases} \tag{5}$$

Then this function is C^1, indeed C^∞, in x, y and t. I have already shown that $\psi(u(x,y),g(t))$ and $\varphi(u(x,y),g(t))$ are C^∞, so the top and bottom lines of Equation 5 are both C^∞, since other components are also C^∞. Now I need to show that their one-sided limits as $x \to 0$ equal the middle line, and that the function is smooth at $x = 0$. For all y, there exists $\varepsilon > 0$ such that as $|x| < \varepsilon$, $u(x,y) < 0$. But as $u < 0$, both $\psi(u(x,y),g(t))$ and $\varphi(u(x,y),g(t))$ are equal to 1, and all of their derivatives are zero. So, all three lines of the piecewise function U coincide, as well as their derivatives, when $x = 0$. Alternatively, one could rewrite Equation 5 as follows:

$$U(x,y;t) = \begin{cases} yx^2 - (e^{-y})\psi(u(x,y),g(t)) & x \leq 0 \\ yx^2 - (e^{-y})\varphi(u(x,y),g(t)) & x \geq 0 \end{cases} \tag{6}$$

Next, $\frac{\partial}{\partial y}U(x,y;t) \neq 0$ for all (x,y,t). To see this, consider U_y:

$$U_y(x,y;t) = \begin{cases} x^2 + (e^{-y})\psi(u(x,y),g(t)) & \\ \qquad -(e^{-y})u_y(x,y)\psi_u(u(x,y),g(t)) & x < 0 \\ x^2 + e^{-y} & x = 0 \\ x^2 + (e^{-y})\varphi(u(x,y),g(t)) & \\ \qquad -(e^{-y})u_y(x,y)\varphi_u(u(x,y),g(t)) & x > 0 \end{cases} \tag{7}$$

The two top lines in Equation 7 are positive, since $\psi_u(u(x,y),g(t)) \leq 0$, $u_y(x,y) > 0$ and $\psi(u(x,y),g(t)) > 0$. Then U_y only needs investigating for

$x > 0$. Rewrite it as follows:

$$x^2 + (e^{-y})\varphi(u(x,y), g(t)) - (e^{-y})u_y(x,y)\varphi_u(u(x,y), g(t)) \\ = \begin{cases} x^2 + e^{-y} & u(x,y) \le 0 \\ x^2 + e^{-y} + (e^{-y})e^{\frac{-g(t)}{u(x,y,)^2}} & \\ \quad -(e^{-y})[x^2 + e^{-y}]\frac{2g(t)}{u(x,y)^3}e^{\frac{-g(t)}{u(x,y,)^2}} & u(x,y) > 0, \end{cases} \tag{8}$$

where the top line is always positive, and the bottom line can be factorised as follows:

$$[x^2 + e^{-y}]\left[1 - (e^{-y})\frac{2g(t)}{u(x,y)^3}e^{\frac{-g(t)}{u(x,y,)^2}}\right] + (e^{-y})e^{\frac{-g(t)}{u(x,y,)^2}} \tag{9}$$

In Equation 9, the first factor and the last part are always positive (in fact, the last part is bounded between 0 and 1, since $u(x,y) > 0$ only for $y > 0$). The second factor turns out to be bounded between 0.180167 and 1.[1]

Hence, U is continuously differentiable and $U_y > 0$ everywhere. However, U has disconnected indifference sets whenever $U(x,y;t) \ge 0$. Further, I will show that a modification of this function satisfies the Spence-Mirrlees condition, but fails to have the single crossing property.

From Equations 3, 4 and 5, it should be clear that $U(x,y,t)$ does not depend on t for $U \le 0$, since for $U \le 0$, $U(x,y,t) \equiv u(x,y)$. The only effect t has, then, comes when $u > 0$ and $U > 0$, through $\psi(\cdot, g(t))$ for $x < 0$ and $\varphi(\cdot, g(t))$ for $x > 0$. For $x < 0$ and $u > 0$, $U_x/|U_y|$ is as follows:

$$\frac{U_x}{|U_y|} = \frac{2yx\left(1 + \frac{2g(t)}{u(\cdot)^3}e^{-g(t)/u(\cdot)^2}e^{-y}\right)}{\left|(x^2 + e^{-y})\left(1 + \frac{2g(t)}{u(\cdot)^3}e^{-g(t)/u(\cdot)^2}e^{-y}\right) - e^{-g(t)/u(\cdot)^2}e^{-y}\right)\right|}, \tag{10}$$

[1]This was found using Mathematica.

while for $x > 0$ and $u > 0$, $U_x/|U_y|$ is as follows:

$$\frac{U_x}{|U_y|} = \frac{2yx\left(1 - \frac{2g(t)}{u(\cdot)^3} e^{-g(t)/u(\cdot)^2} e^{-y}\right)}{\left|(x^2 + e^{-y})\left(1 - \frac{2g(t)}{u(\cdot)^3} e^{-g(t)/u(\cdot)^2} e^{-y}\right) + e^{-g(t)/u(\cdot)^2} e^{-y}\right)\right|}. \tag{11}$$

Differentiating Equations 10 and 11 with respect to t and then simplifying gives, respectively, Equations 12 and 13 below:

$$\begin{aligned}\frac{\partial \frac{U_x}{|U_y|}}{\partial t} = 2e^{-3y-2g(t)/u(x,y)^2} xyg'(t)\Bigg(2e^{2y+g(t)/u(x,y)^2}\Bigg| - e^{-y-g(t)/u(x,y)^2} + \\ +(x^2 + e^{-y})\big(1 + \tfrac{2g(t)}{u(x,y)^3} e^{-y-g(t)/u(x,y)^2}\big)\Bigg| u(x,y)^3(-g(t) + u(x,y)^2) + \\ +\big(2g(t) + e^{y+g(t)/u(x,y)^2} u(x,y)^3\big)\big[2(1 + x^2 e^y)g(t) - \\ -u(x,y)^2\big(2 + 2e^y x^2 + e^y u(x,y)\big)\big]\Bigg)/Q^2,\end{aligned} \tag{12}$$

$$\begin{aligned}\frac{\partial \frac{U_x}{|U_y|}}{\partial t} = 2e^{-3y-2g(t)/u(x,y)^2} xyg'(t)\Bigg(2e^{2y+g(t)/u(x,y)^2}\Bigg| e^{-y-g(t)/u(x,y)^2} + \\ +(x^2 + e^{-y})\big(1 - \tfrac{2g(t)}{u(x,y)^3} e^{-y-g(t)/u(x,y)^2}\big)\Bigg| u(x,y)^3(g(t) - u(x,y)^2) + \\ +\big(-2g(t) + e^{y+g(t)/u(x,y)^2} u(x,y)^3\big)\big[-2(1 + x^2 e^y)g(t) + \\ +u(x,y)^2\big(2 + 2e^y x^2 + e^y u(x,y)\big)\big]\Bigg)/R^2.\end{aligned} \tag{13}$$

In Equation 12, $x < 0$, while in Equation 13, $x > 0$. Every other term in both equations before the large brackets is positive, as are Q^2 and R^2. Hence, to know the sign of the whole equation, further simplifying the inner parts gives:

$$-e^y u(x,y)^5\big(2 + e^{y+g(t)/u(x,y)^2} u(x,y)\big), \tag{14}$$

and

$$e^y u(x,y)^5\big(-2 + e^{y+g(t)/u(x,y)^2} u(x,y)\big). \tag{15}$$

Here, Equation 14 is always negative, which together with $x < 0$ implies that Equation 11 is always positive, for $x < 0$ and $u(x, y) > 0$. In Equation 15, for $x > 0$ and $u(x, y) > 0$, $e^{y+g(t)/u(x,y)^2}u(x, y)$ is bounded below by $e^{g(t)/u(x,y)^2}u(x, y)$ (since $y > 0$), which in turn is bounded below by 2.33164,[2] which means that Equation 15 is always positive, in a given range. This in turn implies that Equation 10 is also always positive, for $x > 0$ and $u(x, y) > 0$. Therefore, by this and everything mentioned earlier, the weak Spence-Mirrlees condition is satisfied everywhere for this function $U(x, y, t)$.

The above function, however, satisfies the single crossing property everywhere. For points in the quadrant I, Theorem 1 applies, as it does for the points in the quadrant IV, while for $y < 0$ the function U does not change in t. However, for all points (x, y) with $x < 0$, U is decreasing in t, while for all $x > 0$, U is increasing. To see that this is the case, note that t enters U only through ψ and φ, which are increasing and decreasing in t, respectively. Since for $x < 0$,

$$\psi_g(\cdot, g(t)) = \frac{g'}{u(x, y)^2}e^{-g(t)/u(x,y)^2} > 0, \tag{16}$$

and hence

$$U_t = -e^{-y}g'\psi_g(\cdot, g(t)) > 0, \tag{17}$$

and for $x > 0$,

$$\varphi_g(\cdot, g(t)) = -\frac{g'}{u(x, y)^2}e^{-g(t)/u(x,y)^2} < 0, \tag{18}$$

and so

$$U_t = -e^{-y}g'\varphi_g(\cdot, g(t)) > 0. \tag{19}$$

Therefore, for all $(\bar{x}, \bar{y})$ and $(\tilde{x}, \tilde{y})$ such that $\bar{x} > 0 > \tilde{x}$, $U(\bar{x}, \bar{y}, t) - U(\tilde{t}, \tilde{y}, t)$ is increasing in t.

[2]This is found using Mathematica, and given that $g(t)$ is bounded between 1 and 2.

We can modify the function U, however, to get a function F with all the desired properties of U, but failing to have the single crossing property. Let $(\bar{x}, \bar{y})$, $(\tilde{x}, \tilde{y})$ and $\tilde{t} < \bar{t}$ be such that $U(\tilde{x}, \tilde{y}, \tilde{t}) = U(\bar{x}, \tilde{y}, \tilde{t})$.

Next, define function q as:

$$q(t) = \frac{U(\tilde{x}, \tilde{y}, t)}{U(\bar{x}, \bar{y}, t)}. \tag{20}$$

Now, let the function $f(U(x, y, t), q(t))$ be defined for $U > 0$ and $t > \tilde{t}$, be bounded below by $U = 0$ for all t, be C^∞ with $\lim_{U \to 0_+} f_U^{(n)}(U(x, y, t), q(t)) = 1$ for all $n = 1, 2, \ldots$, be increasing in U, and be equal to $q(t)U(x, y, t)$ after some $U(x, y, t) = U' < U(\bar{x}, \bar{y}, \tilde{t})$, with $\lim_{U \to U'} f_U^{(n)}(U(x, y, t), q(t)) = 1$. Next, define F as follows:

$$F(x, y, t) = \begin{cases} U(x, y, t) & U(x, y, t) \leq 0 \text{ or } x < 0 \\ f(U(x, y, t), q(t)) & \text{otherwise.} \end{cases} \tag{21}$$

For $U(x, y, t) \leq 0$ or for $x < 0$, $F \equiv U$, so all the properties of U are preserved. For $x > 0$, $0 < U < U'$ and $t > \tilde{t}$, F smoothly changes U into qU, equalling qU for all $U \geq U'$. To see that F preserves the properties of U necessary for the theorem, I will show that F is C^1 and $F_y > 0$.

To see that it is C^1, it suffices to note that f is C^∞ and that for all t, as $U \to 0$ from the right, $f_U \to 1$, and so $f_t = f_U U_t + q' U \to U_t$, $f_y \to U_y$ and $f_x \to U_x$. To show that F is C^∞ is not difficult either.

Next, consider F_y:

$$F_y(x, y, t) = \begin{cases} U_y(x, y, t) & U(x, y, t) \leq 0 \text{ or } x < 0, \\ f_U(U(x, y, t), q(t)) U_y(x, y, t) & \text{otherwise.} \end{cases} \tag{22}$$

Here, both rows are positive, since $f_U > 0$.

Hence, all of the requirements are satisfied. Moreover, F was constructed in such a way as to preserve the shape of the indifference curves of U.

$$\frac{F_x}{|F_y|} = \frac{U_x \times F_U}{|U_y \times F_U|} = \frac{U_x}{|U_y|}, \tag{23}$$

for all (x, y, t), since $F_U > 0$ for all U. Therefore, F satisfies the Spence-Mirrlees condition everywhere.

Now, consider $(\bar{x}, \bar{y})$, $(\tilde{x}, \tilde{y})$ and $\tilde{t} < \bar{t}$ as above, so that $F(\tilde{x}, \tilde{y}, \tilde{t}) = U(\tilde{x}, \tilde{y}, \tilde{t}) = U(\bar{x}, \bar{y}, \tilde{t}) = F(\bar{x}, \bar{y}, \tilde{t})$. But then, as $t = \bar{t}$, $F(\tilde{x}, \tilde{y}, \bar{t}) = U(\tilde{x}, \tilde{y}, \bar{t})$ $= U(\bar{x}, \bar{y}, \bar{t}) \times \frac{U(\tilde{x}, \tilde{y}, \bar{t})}{U(\bar{x}, \bar{y}, \bar{t})} = F(\bar{x}, \bar{y}, \bar{t})$.

Then, there exists some (x^*, y^*) with $0 < x^* < \bar{x}$, such that $F(x^*, y^*, \tilde{t}) > F(\bar{x}, \bar{y}, \tilde{t}) = F(\tilde{x}, \tilde{y}, \tilde{t})$, but $F(x^*, y^*, \bar{t}) < F(\bar{x}, \bar{y}, \bar{t}) = F(\tilde{x}, \tilde{y}, \bar{t})$, as shown in Picture 1. Therefore, F satisfies the Spence-Mirrlees condition, but fails to have the single crossing property.

2 Conclusion

I have shown that, under certain conditions, the result about the equivalence of the (Milgrom-Shannon) single-crossing property and the Spence-Mirrlees single crossing condition only holds for the class of completely regular functions, that is C^1 functions with path-connected indifference sets. If to allow for disconnected indifference sets, even for C^1 functions, the result fails. In particular, the Spence-Mirrlees condition may hold while the single-crossing property fails.

The importance of my finding is that for C^1 functions, in general, comparative statics results which were developed based on the single crossing property may be invalid.

Also, it may be interesting to investigate whether, for such functions, the Spence-Mirrlees condition holds whenever the single-crossing property holds.

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